A reverse Sidorenko inequality

Independent sets, colorings, and graph homomorphisms

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Joint work with

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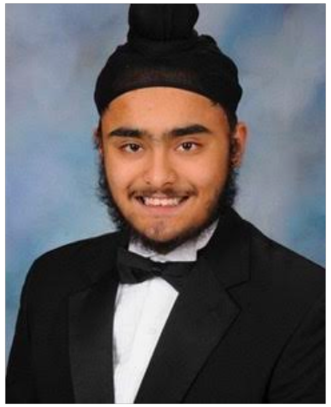

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(MIT)

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Question 1

Fix *d*. Which *d*-regular graph *G* maximizes $i(G)^{1/v(G)}$?

$i(G)$ = the number of independent sets

Question 2

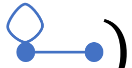
Fix *d* and *q*. Which *d*-regular graph *G* maximizes $c_q(G)^{1/v(G)}$?

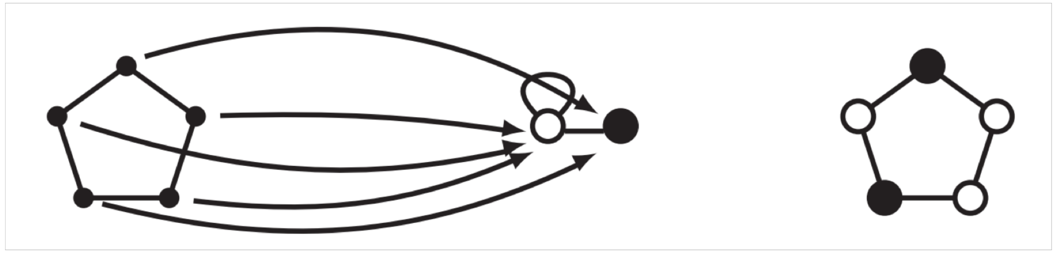
proper *q*-colorings

Question 3

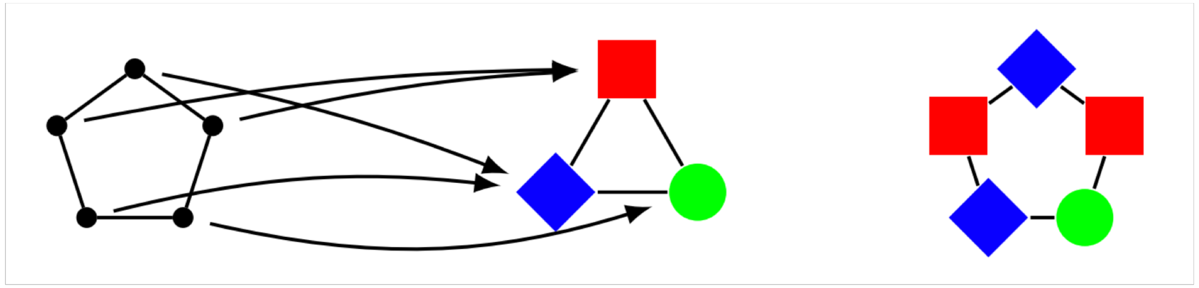
Fix *d* and *H*. Which *d*-regular graph *G* maximizes $\hom(G,H)^{1/v(G)}$?

graph homomorphisms

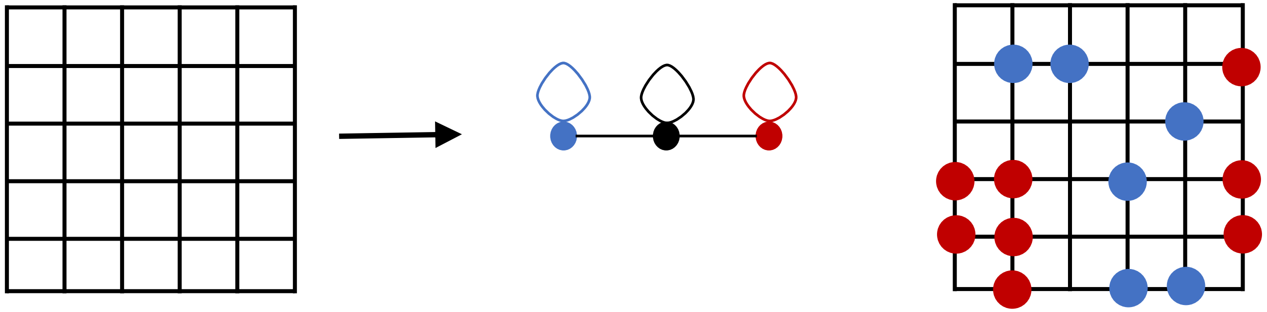
Independent sets: $i(G) = \hom(G,$  $)$



Colorings: $c_q(G) = \hom(G, K_q)$



Widom–Rowlinson model: $\hom(G,$  $)$

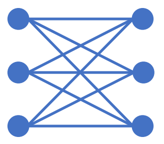


Independent sets

Question 1. Fix *d*. Which *d*-regular graph *G* maximizes $i(G)^{1/v(G)}$?

Asked by Granville in 1988 at Banff in an effort to resolve the Cameron–Erdős conjecture on the number of sum-free subsets of {1, …, *n*}

Conjectured maximizer: $K_{d,d}$



Alon (1991) proved an asymptotic version ($d \to \infty$)

Kahn (2001) proved the conjecture for bipartite *G* via entropy method

Z. (2010) removed the bipartite hypothesis via "bipartite swapping trick" $i(G)^2 \le i(G \times K_2)$

Theorem (Kahn + Z.)**.** Let *G* be an *n*-vertex *d*-regular graph. Then

$$i(G) \le i\left(K_{d,d}\right)^{n/(2d)} = \left(2^{d+1} - 1\right)^{n/(2d)}$$

Davies, Jenssen, Perkins & Roberts (2017) gave a new proof using a novel occupancy method, which found applications in sphere packing and spherical codes [Jenssen, Joos, Perkins 2018]

Graph homomorphisms

Question 3. Fix d and H. Which d-regular graph G maximizes $\hom(G,H)^{1/v(G)}$?

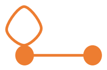
[Galvin, Tetali 2004] Among bipartite graphs, $G = K_{d,d}$ is the maximizer (extending [Kahn '01])

Q. Can the bipartite hypothesis be dropped?

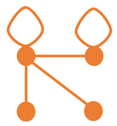
[Z. 2011] Yes for certain families of H, such as threshold graphs (generalizing independent sets).

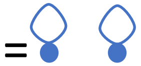
***H* = *K*$_q$ (*q*-colorings) remained open**

1	1
1	0



1	1	1	1
1	1	0	0
1	0	0	0
1	0	0	0



The bipartite hypothesis **cannot** always be dropped. E.g., $H =$ , maximizer is K_{d+1}, not $K_{d,d}$.

[Cohen, Perkins, Tetali 2017] Widom–Rowlinson model ($H =$ ): $G = K_{d+1}$ is the maximizer

[Sernau 2017] $\exists H$: maximizer is neither $K_{d,d}$ nor K_{d+1}

Open: Among 3-regular graphs, is there a finite set of possible maximizers G for $\hom(G,H)^{1/v(G)}$?
(We only know that this set is bigger than $\{K_{3,3}, K_4\}$)

Graph homomorphisms

Question 3. Fix d and H. Which d-regular graph G maximizes $\hom(G, H)^{1/v(G)}$?

Wide open in general (see my survey Extremal regular graphs)

Conjecture (Davies, Jenssen, Perkins, Roberts 2017).
For all fixed H, among triangle-free G, $G = K_{d,d}$ is always the maximizer

(true for bipartite G [Galvin, Tetali 2004])

Independent sets in irregular graphs

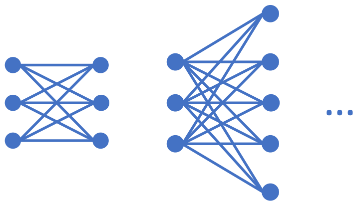
d_u = degree of *u* in G

Degree-degree distribution: probab. distribution of (d_u, d_v) for uniformly random edge *uv*

Question 1'. Given the degree-degree distribution, which *G* maximizes $i(G)^{1/v(G)}$?

e.g., 20% edges have endpoint degrees (3,4), 30% edges …

Conjecture (Kahn '01)**.** Maximizer is a disjoint union of complete bipartite graphs



We prove this conjecture

Theorem (Sah, Sawhney, Stoner, Z., '18+)**.** Let *G* be a graph without isolated vertices. Then

$$i(G) \le \prod_{uv \in E(G)} i\left(K_{d_u,d_v}\right)^{1/(d_u d_v)}$$

Independent sets are **biclique-maximizing**

Conjecture (Galvin '06)**.** An analogous inequality for $\hom(G, H)$ (False; which *G* and *H*?)

Proper colorings

Question 2. Fix d and q. Which d-regular graph G maximizes $c_q(G)^{1/v(G)}$?

Conjectured answer: $K_{d,d}$

[Galvin, Tetali '04] True for bipartite G

[Davies, Jenssen, Perkins, Roberts '18] True for $d = 3$ & [Davies] $d = 4$ (computer-assisted)

We prove the conjecture

Theorem (Sah, Sawhney, Stoner, Z. '18++)**.** Let $q \in \mathbb{N}$ and G an n-vertex d-regular graph. Then

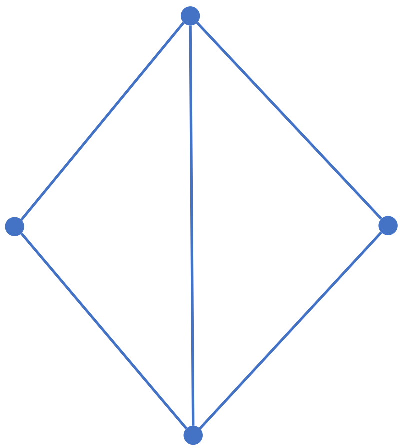
$$c_q(G) \le c_q\left(K_{d,d}\right)^{n/(2d)}$$

Theorem (Sah, Sawhney, Stoner, Z.)**.** Let $q \in \mathbb{N}$ and G a graph without isolated vertices. Then

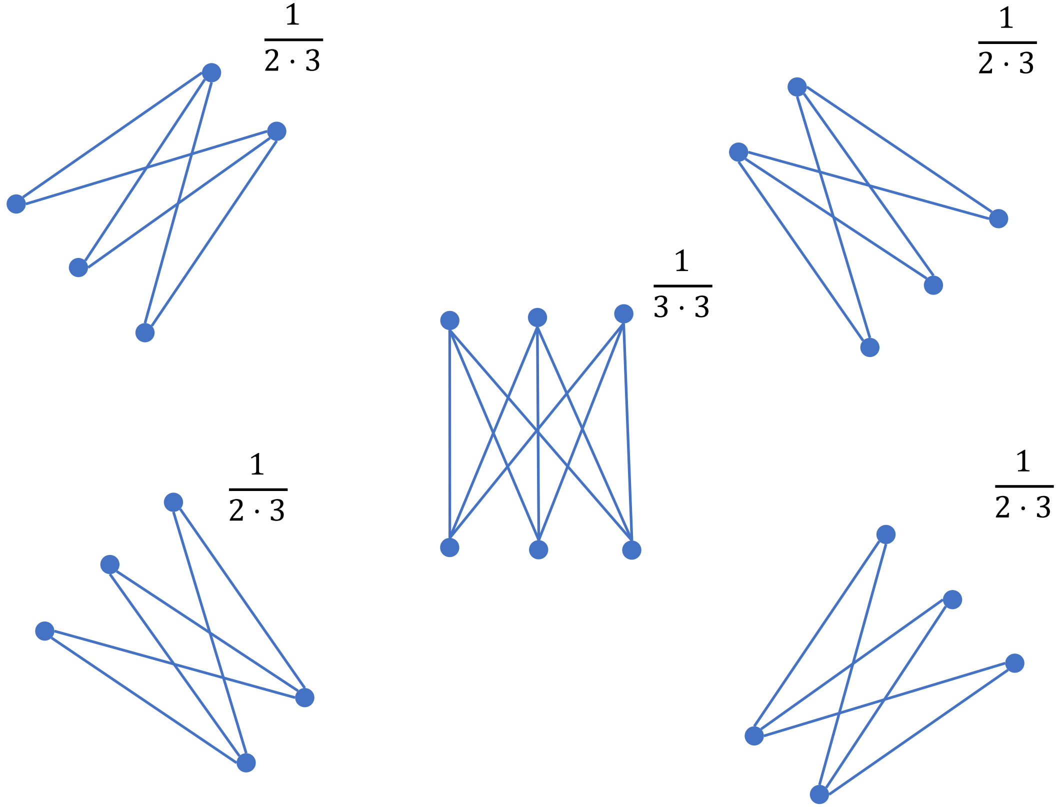
$$c_q(G) \le \prod_{uv \in E(G)} c_q\left(K_{d_u,d_v}\right)^{1/(d_u d_v)}$$

Proper colorings are **biclique-maximizing**

The number of independent sets and proper q-colorings satisfies



$\leq$



$$f(G) \leq \prod_{uv \in E(G)} f\left(K_{d_u, d_v}\right)^{1/(d_u d_v)}$$

f counts independent sets or proper q-colorings

Graph homomorphisms

Question 3. Fix d and H. Which d-regular graph G maximizes $\hom(G,H)^{1/v(G)}$?

Conjecture (Davies, Jenssen, Perkins, Roberts '17). Among triangle-free G, $G = K_{d,d}$ is always the maximizer (already known for bipartite G [Galvin, Tetali '04])

We prove this conjecture

Theorem (Sah, Sawhney, Stoner, Z.). Let G be a triangle-free n-vertex d-regular graph. Then

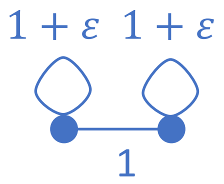
$$\hom(G,H) \le \hom(K_{d,d},H)^{n/(2d)}$$

Theorem (SSSZ). Let G be a triangle-free graph without isolated vertices. Then

$$\hom(G,H) \le \prod_{uv\in E(G)} \hom(K_{d_u,d_v},H)^{1/(d_u d_v)}$$

Always **biclique-maximizing** among triangle-free graphs

False for every G with a triangle! Counterexample: $H = \begin{pmatrix} 1+\varepsilon & 1 \\ 1 & 1+\varepsilon \end{pmatrix}$ as $\varepsilon \to 0$



Reverse Sidorenko inequality

Sidorenko's conjecture: for bipartite G, all H

$$t(G,H) \geq t(K_2,H)^{e(G)}$$

$t(G,H) = \hom(G,H)/v(H)^{v(G)}$

[Hatami] [Conlon, Fox, Sudakov] [Li, Szegedy] [Kim, Lee, Lee] [Conlon, Kim, Lee, Lee] [Szegedy] [Conlon, Lee]

Open for $G = K_{5,5} \setminus C_{10}$ (Möbius strip)

Our result: for triangle-free d-regular G

$$t(G,H) \leq t\left(K_{d,d},H\right)^{e(G)/d^2}$$

$\|\cdot\|_G := t(G,\ \cdot\ )^{1/e(G)}$ (Hatami's graph "norm"; [Conlon, Lee]). For graphon $W\colon [0,1]^2 \to [0,1]$,

$$\|W\|_{K_2} \leq \|W\|_G \leq \|W\|_{K_{d,d}}$$

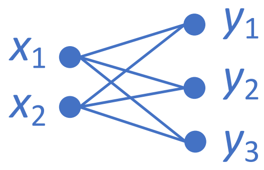
bipartite G (Sidorenko's conjecture) ↗ ? triangle-free d-regular G (our result) ↖

Theorem (Sah, Sawhney, Stoner, Z.)**.** Let G be a triangle-free graph and $W\colon [0,1]^2 \to [0,1]$. Then

$$t(G,W) \leq \prod_{uv \in E(G)} \|W\|_{K_{d_u,d_v}}$$

Reverse Sidorenko inequality

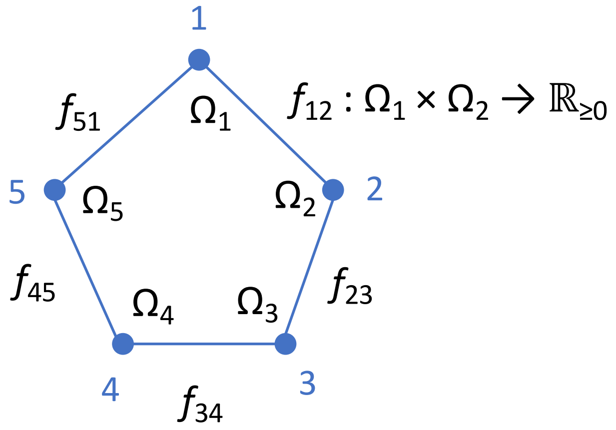
Given $f: \Omega_1 \times \Omega_2 \to \mathbb{R}$, e.g., $\|f\|_{K_{2,3}} =$



$$\left| \int_{\Omega_1^2 \times \Omega_2^3} f(x_1, y_1) f(x_1, y_2) f(x_1, y_3) f(x_2, y_1) f(x_2, y_2) f(x_2, y_3)\, dx_1 dx_2 dy_1 dy_2 dy_3 \right|^{1/6}$$

Theorem (Sah, Sawhney, Stoner, Z.).
Triangle-free graph $G = (V, E)$ without isolated vertices, $f_{uv} \geq 0$,

$$\int \prod_{uv \in E} f_{uv}(x_u, x_v)\, d\boldsymbol{x}_V \leq \prod_{uv \in E} \|f_{uv}\|_{K_{d_v, d_u}}$$



Reverse Sidorenko inequality

Theorem (Sah, Sawhney, Stoner, Z.)**.**
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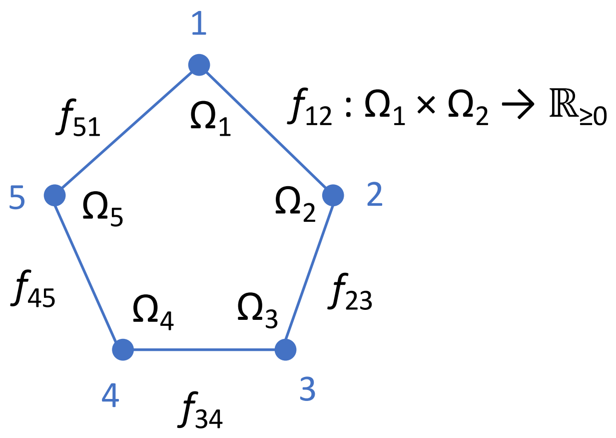

Reverse Sidorenko inequality

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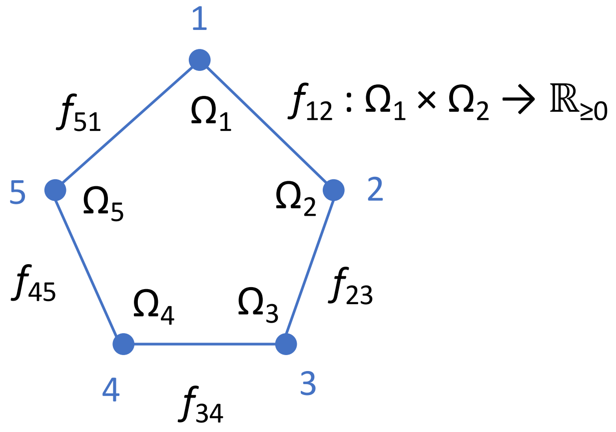


Graphical analogs of Brascamp—Lieb type inequalities:

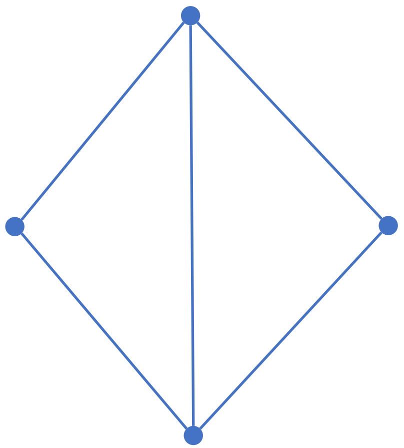
$$\int f_1(\dots) \dots f_k(\dots) \lesssim \|f_1\|_{L^{p_1}} \dots \|f_k\|_{L^{p_k}}$$

Note that (by Hölder)

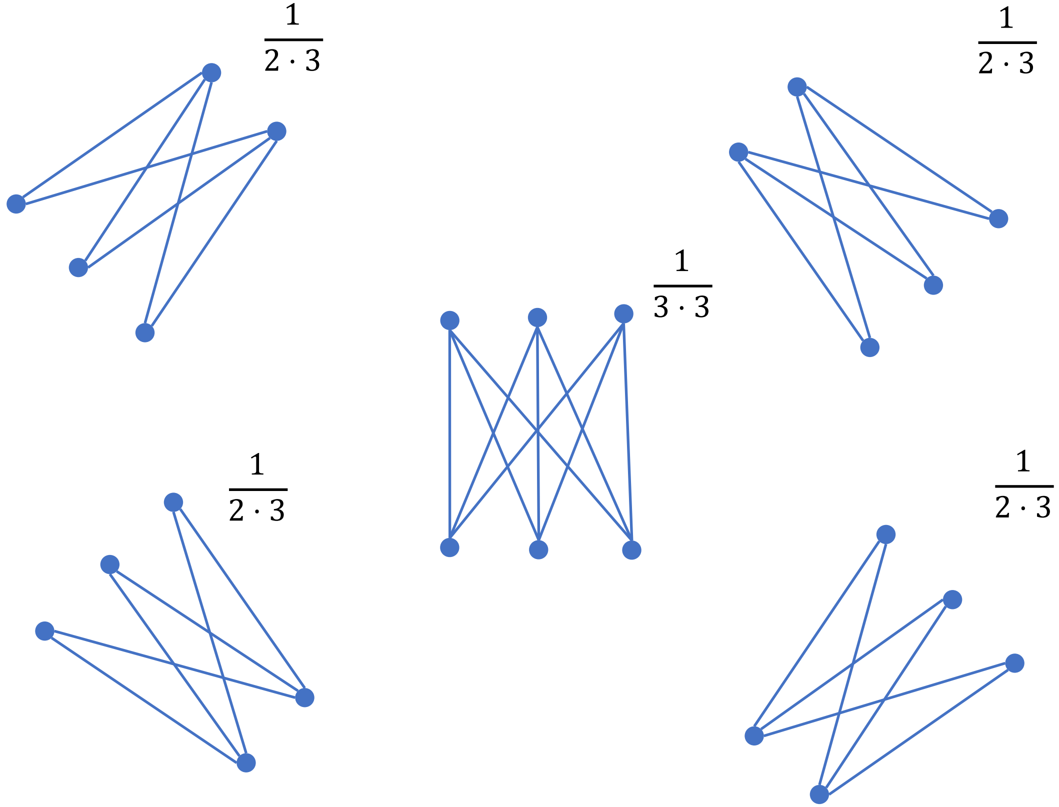
$$\|f\|_{K_{a,b}} \leq \|f\|_{L^{ab}}$$

Future direction: extensions to simplicial complexes

The number of independent sets and proper q-colorings



$\leq$

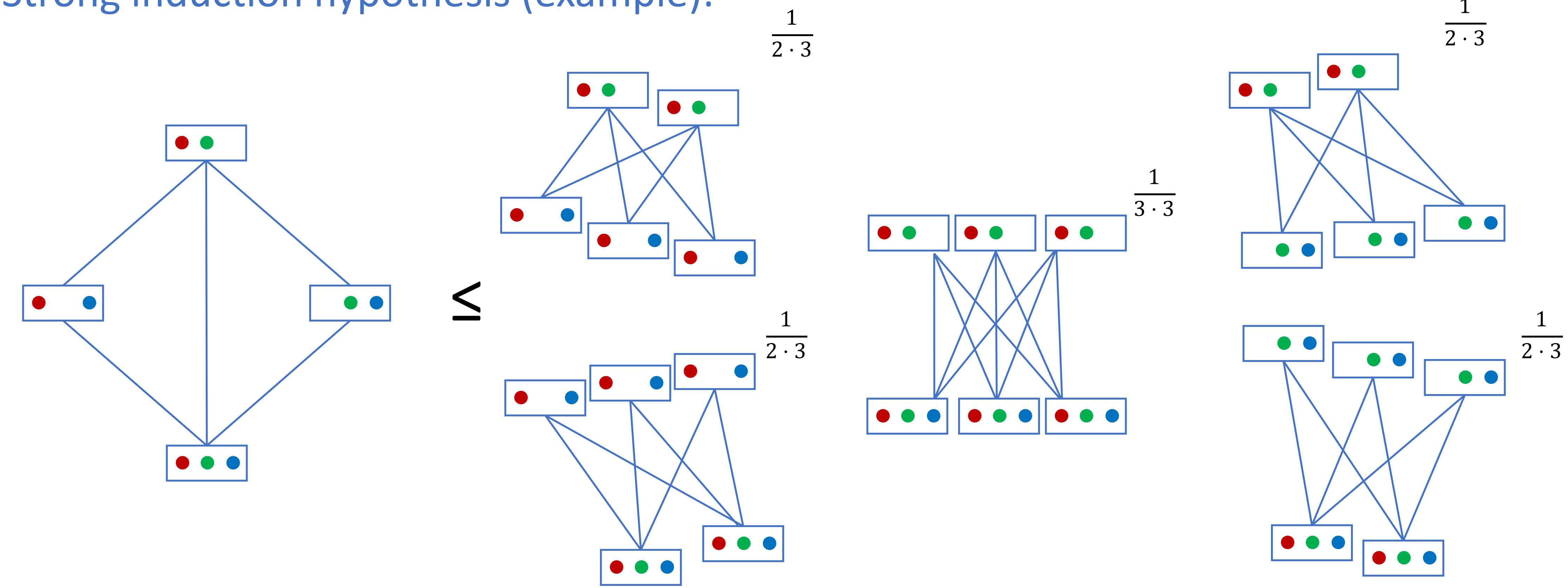


$$f(G) \leq \prod_{uv \in E(G)} f\left(K_{d_u,d_v}\right)^{1/(d_u d_v)}$$

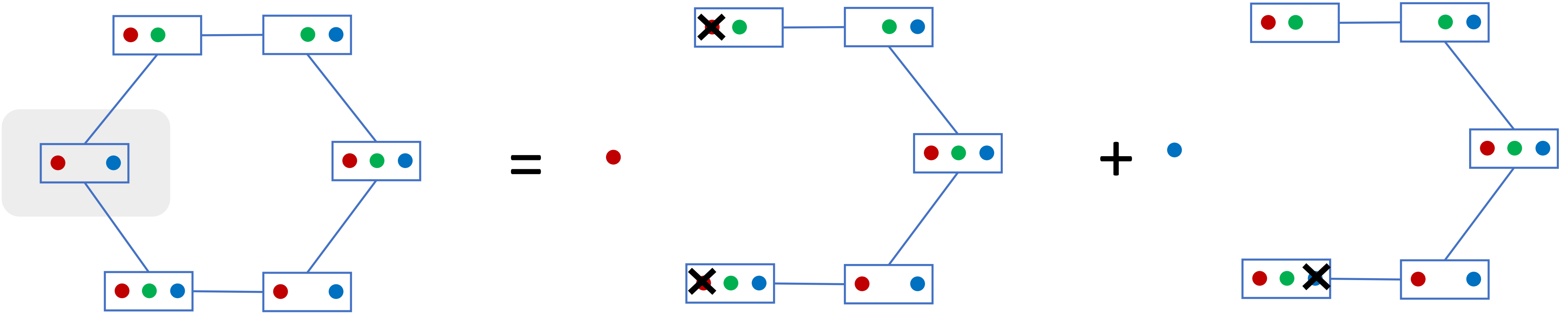
f counts independent sets or proper q-colorings

The number of proper list colorings

Strong induction hypothesis (example):

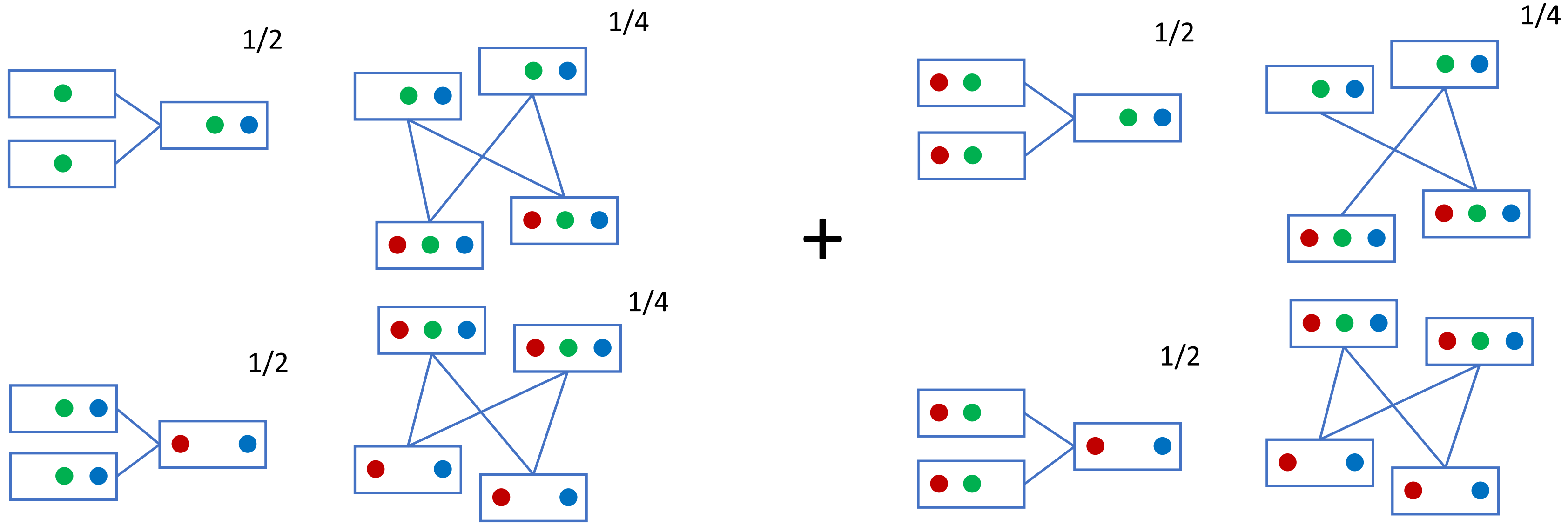


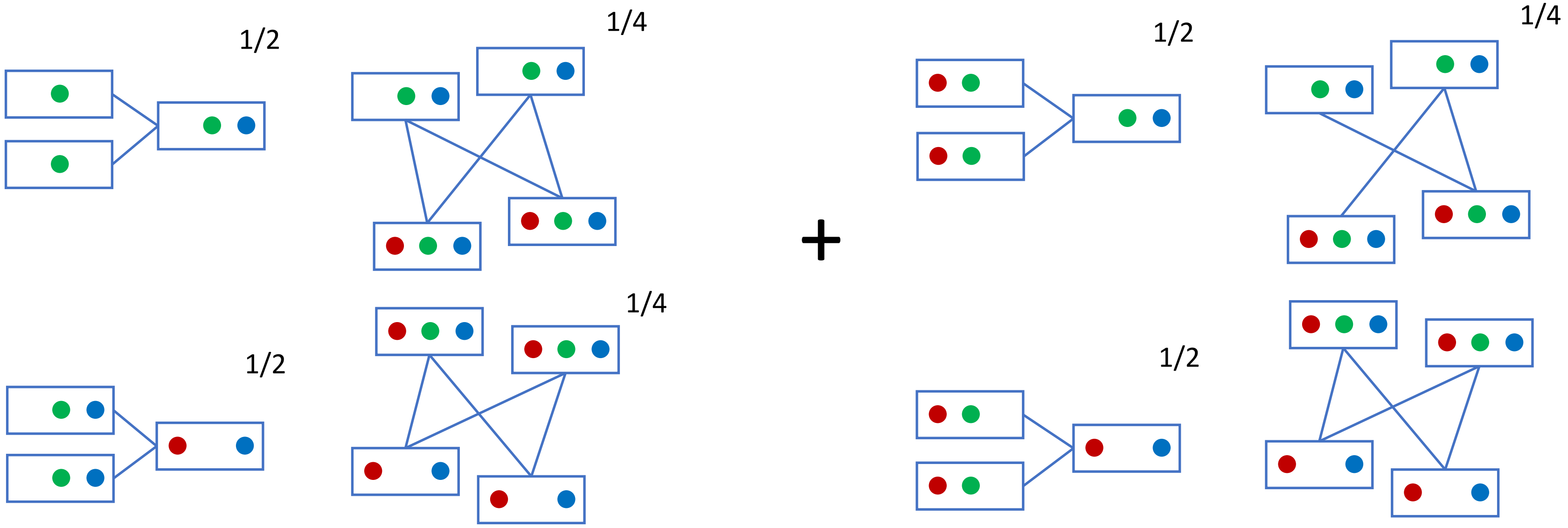
Proof strategy: Induction



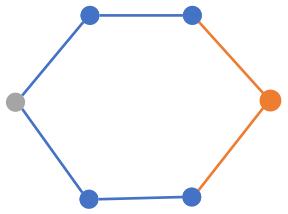
By induction

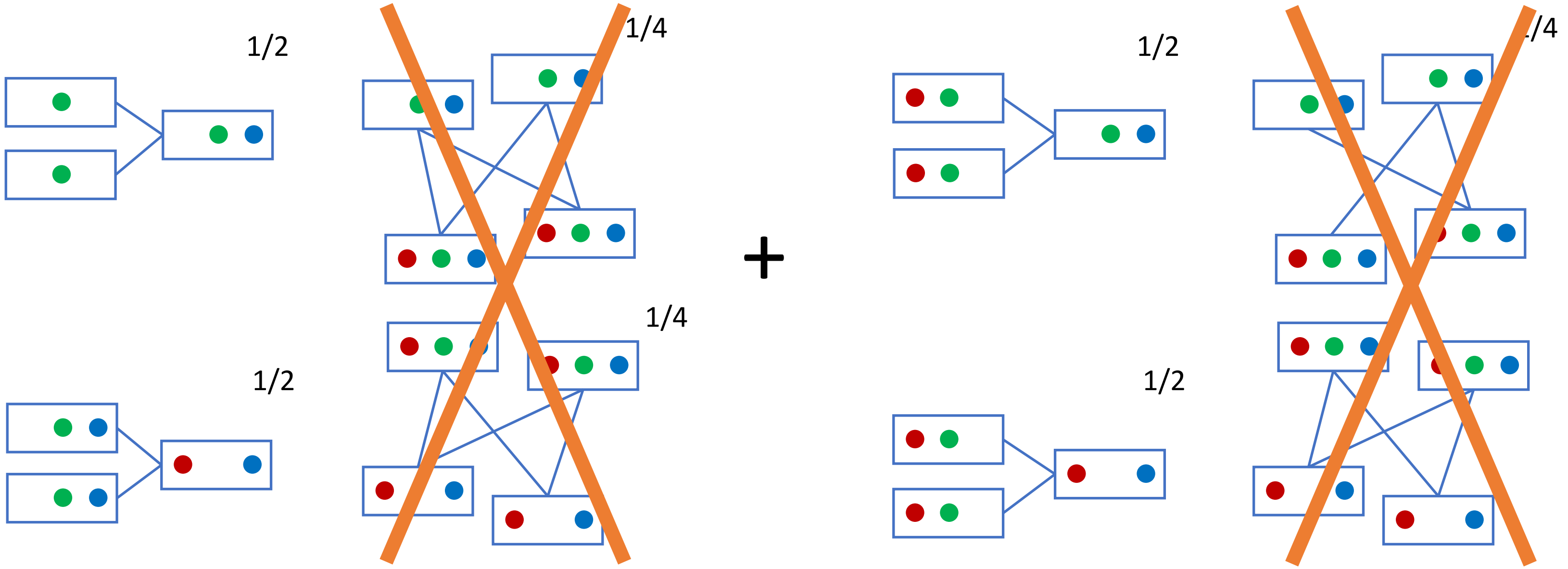
≤




1/2
1/4
1/4
1/2
+
1/2
1/4
1/2

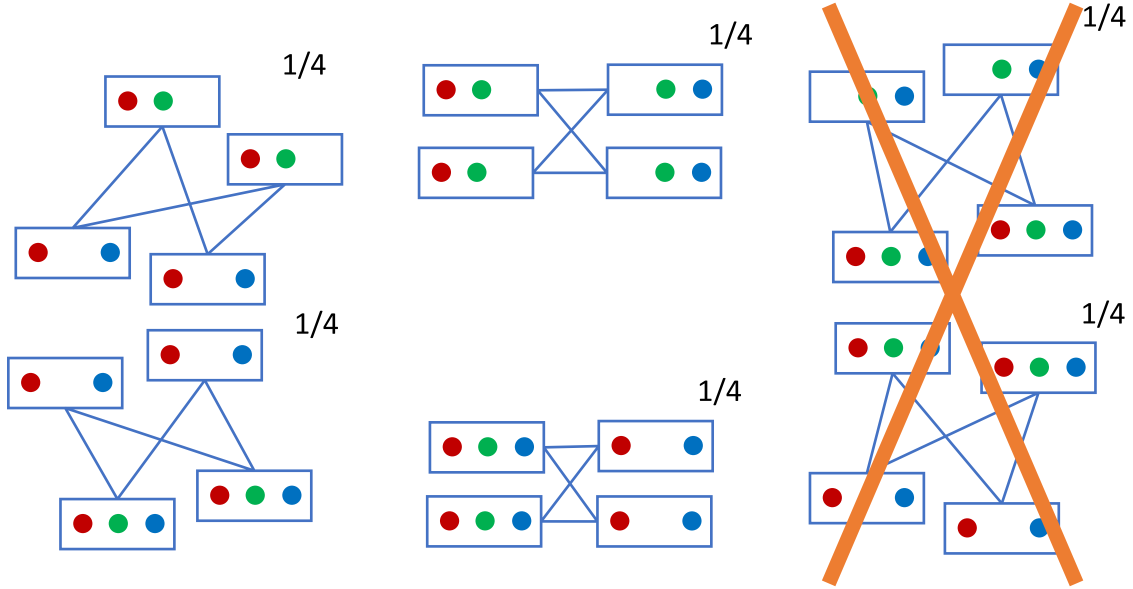
Proof strategy: Reduction to local inequality



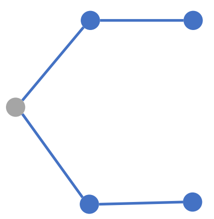


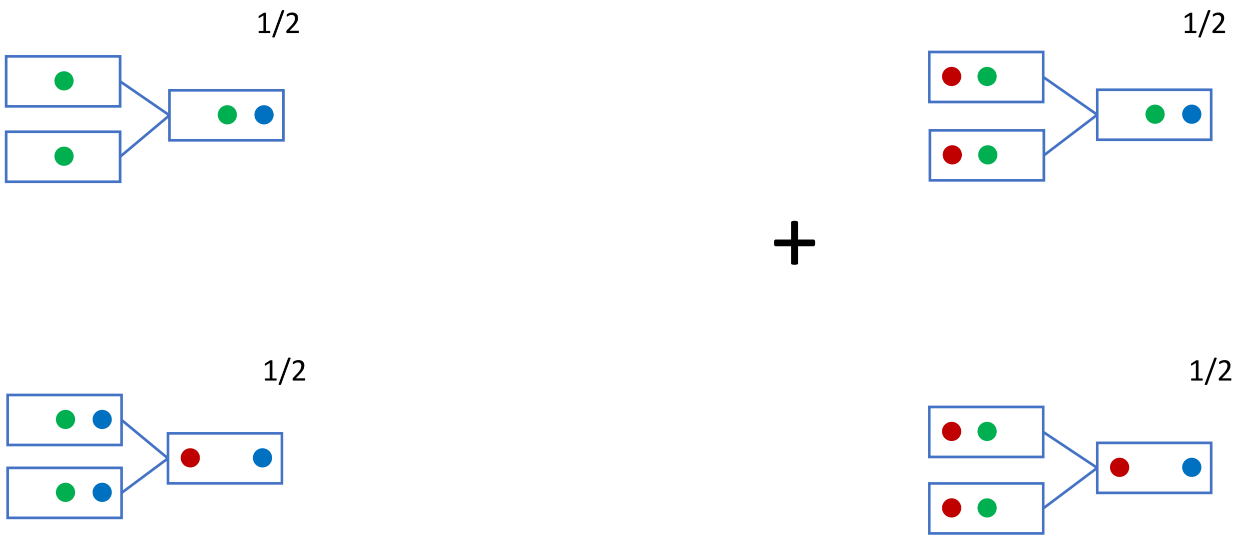
Remains to show

≤



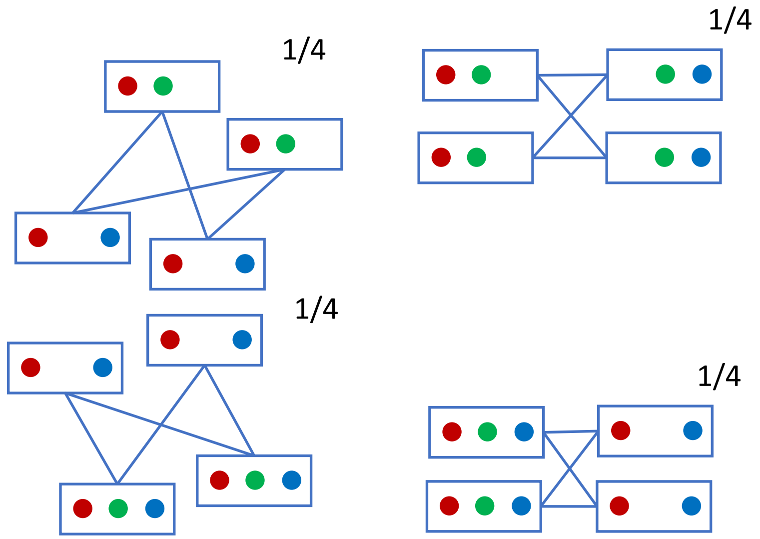
Proof strategy: Reduction to local inequality



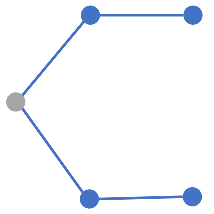


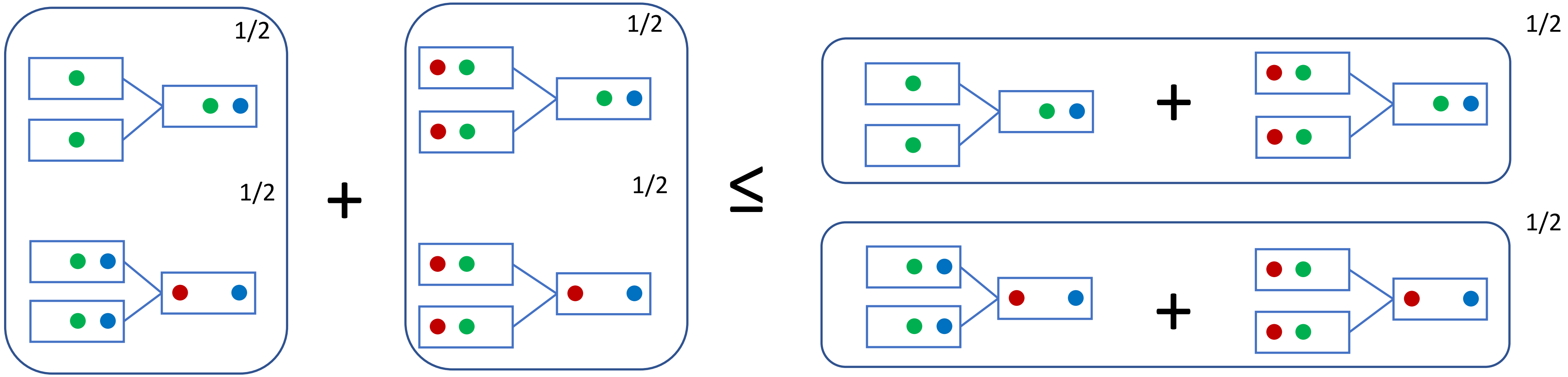
Remains to show

≤



Proof strategy: Reduction to local inequality

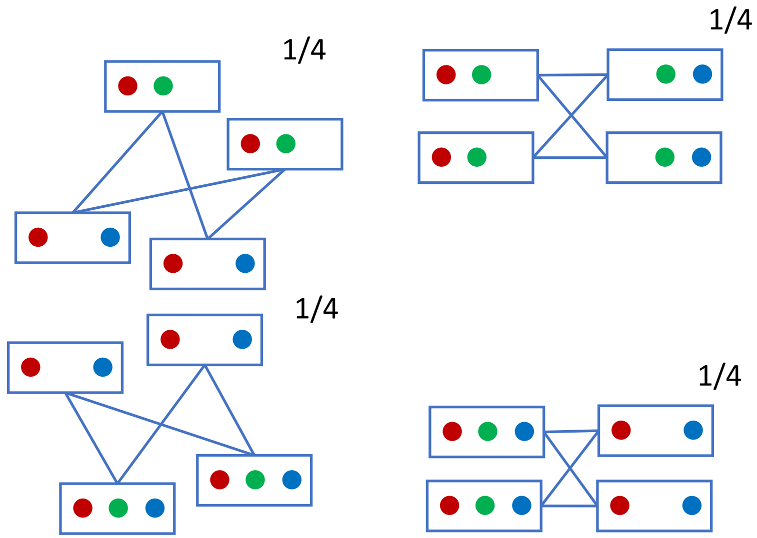




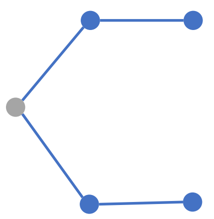
By Cauchy—Schwarz: $\sqrt{AB} + \sqrt{CD} \leq \sqrt{(A+C)(B+D)}$

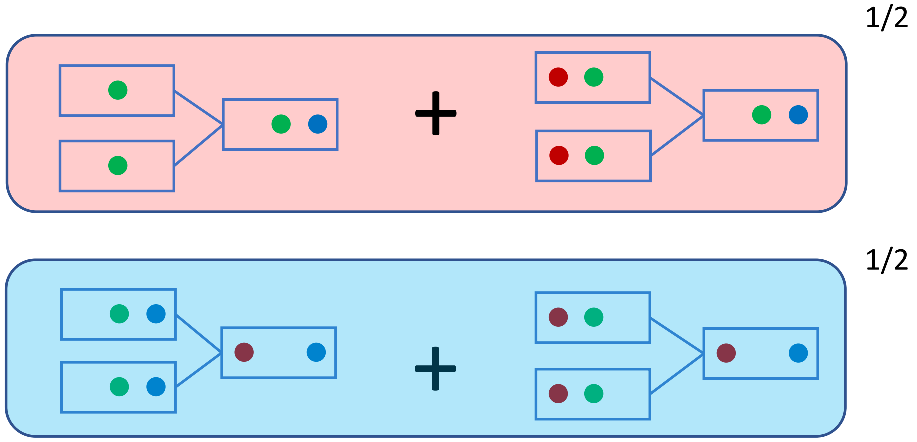
Remains to show

≤



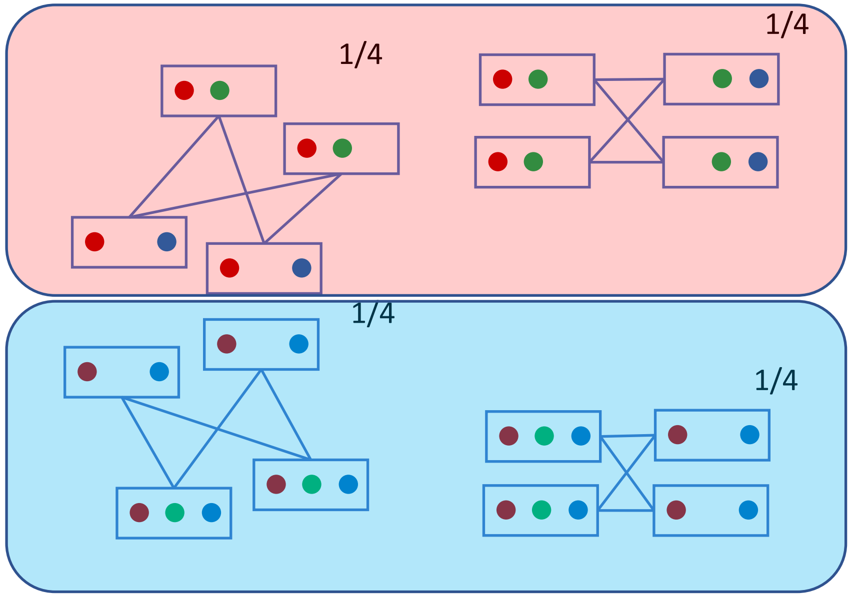
Proof strategy: Reduction to local inequality





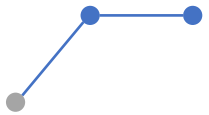
Remains to show

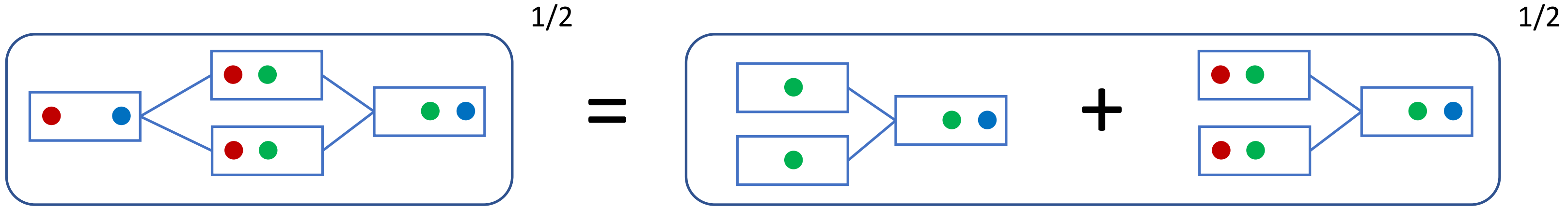
≤



Break inequality into two parts:
top & bottom

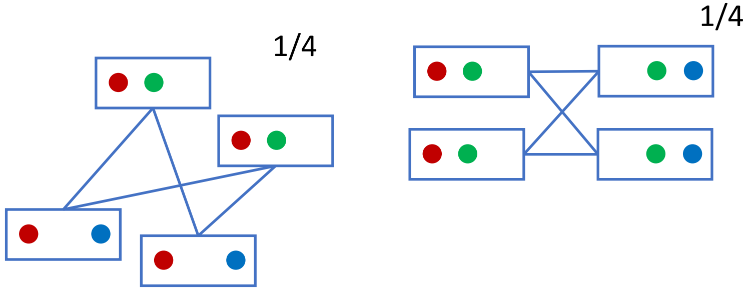
Proof strategy: Local inequality





Remains to show

≤



Break inequality into two parts:
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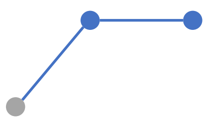
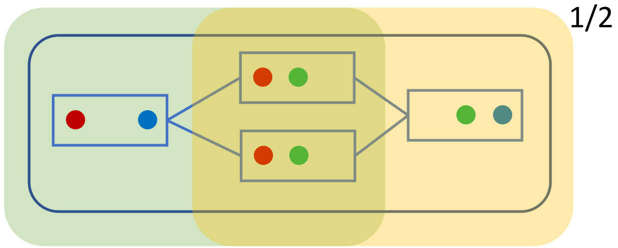
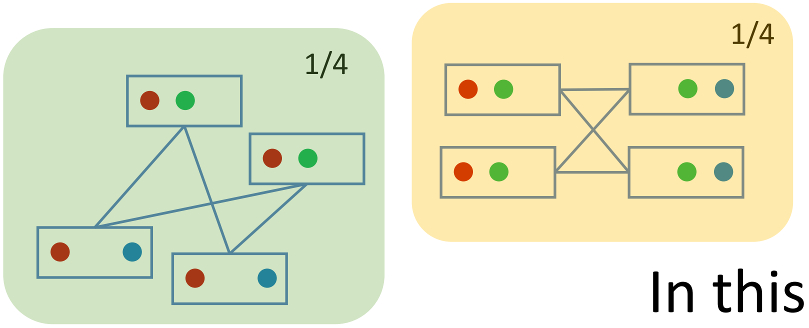
Proof strategy: Local inequality





This is a minimal instance of the inequality

Remains to show



≤

In this case, follows from Cauchy—Schwarz

Much more difficult if *G* has triangles
(not always true for other models!)

A useful matrix inequality

Define the mixed $\ell^{p,q}$ norm of matrix $A = (a_{ij})$ by first taking ℓ^p norm of each row, and then taking ℓ^q norm of the results, i.e.

$$\|A\|_{p,q} := \left(\sum_i \left(\sum_j |a_{ij}|^p\right)^{q/p}\right)^{1/q}$$

Lemma. For positive semidefinite (PSD) matrix A with nonneg entries, and $q \geq 1$,

$$\|A\|_{1,q}^2 \leq \|A\|_{1,1} \|A\|_{q,q}$$

Question. Is it true that for all $1 \leq p \leq q$,

$$\|A\|_{p,q}^2 \leq \|A\|_{p,p} \|A\|_{q,q} \ ?$$

Graph homomorphisms

Question 3. Fix *d* and *H*. Which *d*-regular graph *G* maximizes $\hom(G,H)^{1/v(G)}$?

Let *H* be a nonneg weighted graph (model)
$\hom(G,H)$ = partition function of some stat. phys. model, e.g., hard-core, Ising, Potts. Say:

- *H* is biclique-maximizing if $Z(G) := \hom(G,H)$ satisfies

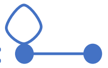
$$Z(G) \le \prod_{uv\in E(G)} Z\left(K_{d_u,d_v}\right)^{1/(d_u d_v)}$$

i.e., conditioned on degree-degree distribution

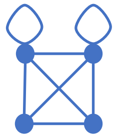
- *H* is clique-maximizing if $Z(G) := \hom(G,H)$ satisfies

$$Z(G) \le \prod_{v\in V(G)} Z\left(K_{d_v+1}\right)^{1/(d_v+1)}$$

i.e., conditioned on degree distribution

Our results: *H* =  (indep sets) and K_q (proper colorings) are both biclique-maximizing

More generally, every partially looped K_q (semiproper colorings) is biclique-maximizing



Ferromagnetism and anti-ferromangnetism

Given a nonneg weighted graph/model *H*, we say that

- *H* is ferromagnetic if its edge-weight matrix is positive semidefinite, i.e., all eigenvalues are nonnegative: $0 \le \cdots \le \lambda_3 \le \lambda_2 \le \lambda_1$ (e.g., *H* = )
- *H* is antiferromagnetic if its edge-weight matrix has at most one positive eigenvalue: $\cdots \le \lambda_3 \le \lambda_2 \le 0 \le \lambda_1$ (e.g., indep sets and colorings)

Theorem (Sah, Sawhney, Stoner, Z.). Every ferromagnetic model is clique-maximizing

Conjecture 1. Every clique-maximizing model is ferromagnetic

Conjecture 2. Every antiferromagnetic model is biclique-maximizing

Our results verify Conj. 2 for independent sets and colorings. Open for Potts model

Widom—Rowlingson model is clique-maximizing among *d*-regular *G* [Cohen, Perkins, Tetali] but not for irregular *G*, and it is not ferromagnetic. 

Two-spin systems

- An Ising model with nonneg edge-weight matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is ferromagnetic if $ac \geq b^2$ and antiferromagnetic if $ac \leq b^2$
  E.g., independent set $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is antiferromagnetic

Corollary (Sah, Sawhney, Stoner, Z.)**.** A 2-spin model is

- Biclique-maximizing if antiferromagnetic, and
- Clique-maximizing if ferromagnetic

This generalizes the result for independent sets

A similar classification for 3-spin systems is open

Summary of main results

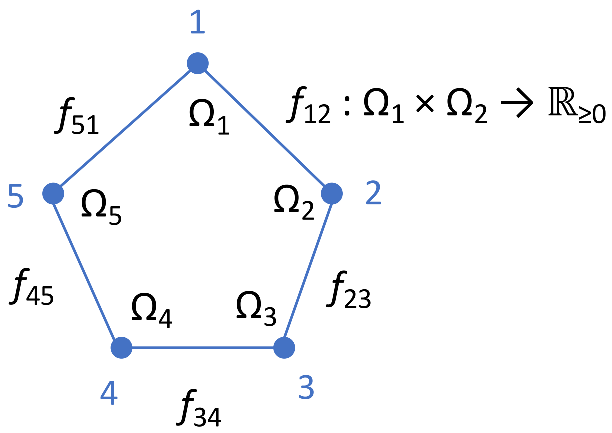
- Independent sets and proper colorings are biclique-maximizing
- Every ferromagnetic model is clique-maximizing
- Every model is biclique-maximizing when restricted to triangle-free graphs

Reverse Sidorenko inequality. (Sah, Sawhney, Stoner, Z.).
Triangle-free graph $G = (V, E)$ without isolated vertices, $f_{uv} \geq 0$,

$$\int \prod_{uv \in E} f_{uv}(x_u, x_v)\, d\boldsymbol{x}_V \leq \prod_{uv \in E} \|f_{uv}\|_{K_{d_v,d_u}}$$

Corollary. For triangle-free G without isolated vertices, $\forall\, H$

$$\hom(G, H) \leq \prod_{uv \in E(G)} \hom(K_{d_u,d_v}, H)^{1/(d_u d_v)}$$



Conjecture. Every antiferromagnetic model is biclique-maximizing (e.g., Potts).