Popular common difference

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Based on joint works arXiv:1903.04863, 2004.07722, 2004.13690
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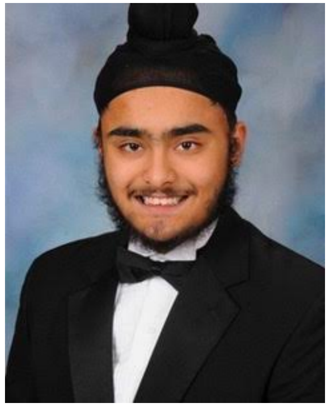

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Roth's theorem. Fix $\alpha > 0$. For all sufficiently large N, every subset of $[N] := \{1,2,\ldots,N\}$ with $\geq \alpha N$ elements contains a 3-AP

A random subset of $\mathbb{Z}/N\mathbb{Z}$ with density α has $\approx \alpha^3$ fraction of all 3-APs

There exists $A \subset \mathbb{Z}/N\mathbb{Z}$ with $|A| \geq \alpha N$ and whose #3-AP is $\leq \alpha^{c \log(1/\alpha)} N^2$ much less than $\alpha^3 N^2$ (a blow-up of Behrend construction)

Nonetheless, can find "popular common difference" that is at roughly as least as popular as random:

Roth's theorem with popular difference (Green '05). $\forall\, \epsilon > 0\ \exists\, N_0(\epsilon)$ so that for all $N \geq N_0(\epsilon)$ and $A \subset [N]$ with $|A| \geq \alpha N$, $\exists\, d \neq 0$ such that the number of 3-APs in A with common difference d is $\geq (\alpha^3 - \epsilon)N$

Proved via an arithmetic analog of Szemerédi's graph regularity lemma

Roth's theorem with popular difference (Green '05). $\forall\, \epsilon > 0\ \exists\ N_0(\epsilon)$ so that for all $N \geq N_0(\epsilon)$ and $A \subset [N]$ with $|A| \geq \alpha N$, $\exists\ d \neq 0$ such that the number of 3-APs in A with common difference d is $\geq (\alpha^3 - \epsilon)N$

How large does $N_0(\epsilon)$ need to be?

Arithmetic regularity lemma needed $\mathrm{tower}(\epsilon^{-O(1)}) = 2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}$ (height $\epsilon^{-O(1)}$),

tight for the regularity lemma [Gowers, Green, Hosseini—Lovett–Moshkovitz–Shapira]

Theorem (Fox—Pham—Z.). The optimal $N_0(\epsilon)$ is $\mathrm{tower}(\Theta(\log 1/\epsilon))$

(Extends earlier results of Fox—Pham for finite field setting $\mathbb{F}_p^n$)

First application of regularity method where tower-type bounds are necessary

What about patterns other than 3-APs?

Multidimensional patterns in $\mathbb{Z}^d$?

Szemerédi's 4-AP theorem with popular difference (Green–Tao '10). $\forall\ \epsilon > 0$ $\exists\ N_0(\epsilon)$ so that for all $N \geq N_0(\epsilon)$ and $A \subset [N]$ with $|A| \geq \alpha N$, $\exists\ d \neq 0$ such that the number of 4-APs in A with common difference d is $\geq (\alpha^4 - \epsilon)N$

Proof uses quadratic Fourier analysis

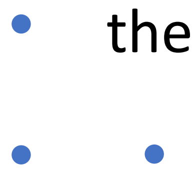
Theorem (Bergelson–Host–Kra–Ruzsa 2005). False for k-AP for $k \geq 5$

What about other patterns?

Whereas 4-APs are dilation of {0,1,2,3}, what about dilations of {0,1,2,4}?

Patterns in $\mathbb{Z}^d$? (Pattern = dilation of a fixed set)

(Note: no applicable higher order Fourier analysis for multidimensional Szemerédi theorem)



"corner"

Patterns with popular difference property

Question. Which finite sets $P \subseteq \mathbb{Z}^r$ (with ≥ 3 points) have the following property:
$\forall A \subseteq [N]^r \ \exists d \neq 0 : A$ contains $\left(\alpha^{|P|} - o(1)\right) N^r$ translates of $d \cdot P = \{dp : p \in P\}$,
where $\alpha = |A|/N^r$?

Green: all 3-point patterns

Green–Tao: 4-APs. More generally, 4-point patterns of the form $\{0, a, b, a + b\}$

Theorem (Sah–Sawhney–Z.). No other patterns have the popular diff. property

Question. For each given pattern, what "popular difference density" can you guarantee?

Green's proof of Roth with popular difference

Illustrated here for $\mathbb{F}_3^n$ (more generally, use Bohr sets instead of subspaces)

Let $A \subset \mathbb{F}_3^n$ and $f = 1_A$

Regularity lemma. ∃subspace H with codim $\text{tower}(\epsilon^{-O(1)})$ so that

f_H approximates f really well

(Here f_H is obtained by averaging f along each H-coset; i.e., $f_H = f * \mu_H$)

Denote the density of 3-APs with common difference in H by

$$\Lambda_H(f) \coloneqq \mathbb{E}_{x\in\mathbb{F}_3^n,\, y\in H} f(x)f(x+y)f(x+2y)$$

Counting lemma. $\Lambda_H(f) \geq \Lambda_H(f_H) - \epsilon = \mathbb{E}[f_H^3] - \epsilon \geq (\mathbb{E}f)^3 - \epsilon$

$f_H(x) = f_H(x+y)$
$= f_H(x+2y)\ \forall y \in H$

Thus, provided H not too small, one can find a popular common difference.
Proof works provided $n \geq \text{tower}(\epsilon^{-c})$

A more efficient “cubic energy” increment

(Fox—Pham, Fox—Pham—Z.) Use “cubic energy” $\mathbb{E}[f_H^3]$

Regularity lemma. ∃ subspaces $W \le U \le \mathbb{F}_3^n$ with codim $\mathrm{tower}\big(O(\log 1/\epsilon)\big)$ such that

$$\|(f - f_W)^\wedge\|_\infty \le \frac{\epsilon}{|U^\perp|} \quad \text{and} \quad 2\|f_U\|_3^3 - \|f_W\|_3^3 \ge (\mathbb{E}f)^3 - \epsilon$$

Energy roughly doubles at each iteration of the regularity proof; $O(\log 1/\epsilon)$ *iterations*

Schur’s inequality. $a^3 + b^3 + c^3 + 3abc \ge a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2 \quad \forall a, b, c \ge 0$

Deduce that ∀ subspaces $W \le U$, $\quad \Lambda_U(f_W) \ge 2\|f_U\|_3^3 - \|f_W\|_3^3$

Counting lemma. With $W \le U$ produced by the regularity lemma,

$$\Lambda_U(f) \ge \alpha^3 - O(\epsilon)$$

Thus if U is not too small, then it contains a nonzero popular common difference.

Proof works provided $n \ge \mathrm{tower}(c \log 1/\epsilon)$

Ideas of lower bound constructions

[Gowers '97] Tower-type bounds are necessary for Szemerédi's regularity lemma.

Iterative construction.

[Fox—Pham—Z.] Construction of $f\colon \mathbb{Z}/N\mathbb{Z} \to [0,1]$ without 3-AP popular difference

Let $H_i = \mathbb{Z}/p_i\mathbb{Z}$, with primes $p_1 \ll p_2 \ll \cdots \ll p_s$. Construct weighted functions iteratively

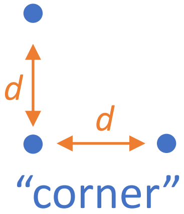
$f_1\colon H_1 \to [0,1]$ $\quad$ $f_2\colon H_1 \times H_2 \to [0,1]$ $\quad$ … $\quad$ $f_s\colon H_1 \times \cdots \times H_s \to [0,1]$

Each f_i is obtained from the previous by extending + random modification,

Maintain the property that each f_i has the 3-AP popular difference

Finish by a "Behrend-twist"

Popular difference result for corners?



Question. Given $A \subset [N]^2$ with $|A| \geq \alpha N^2$, is there always some nonzero $d \neq 0$ such that there are $\geq (\alpha^3 - o(1))N^2$ corners with common difference d, i.e., $(x,y), (x + d, y), (x, y + d) \in A$?

Matei Mandache: No! Construction with $< \alpha^{3.13}N^2$ corners for every common difference

On the other hand, can find d with $\geq (\alpha^4 - o(1))N^2$ corners
(Mandache proved it for $\mathbb{F}_2^n$; extended to abelian groups and intervals by Aaron Berger)

Reduces the problem to a certain variational problem

- Upper bound (probabilistic construction)
- Lower bound (arithmetic regularity for corners; Fourier for corners due to Shkredov)

Theorem (Fox—Sah—Sawhney—Stoner—Z.). The optimal popular difference density that can be guaranteed for corners is $\geq \omega(\alpha^4)$ and $\leq \alpha^{4-o(1)}$

Note: everything applies to patterns of 3 non-collinear points in $\mathbb{Z}^2$

The triforce

Triforce and corners

Mandache/Berger reduce the popular difference problem for corners to

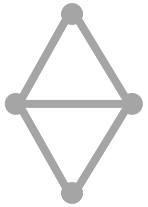
Problem. Find the minimum triforce density $g(\alpha)$ in 3-uniform a hypergraph with triple density α

Theorem (Fox—Sah—Sawhney—Stoner—Z.). $\omega(\alpha^4) \le g(\alpha) \le \alpha^{4-o(1)}$

Lower bound: triangle removal lemma

Upper bound: Behrend construction

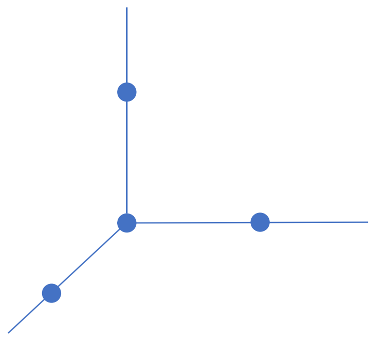
More generally, the maximum *k*-force density in a *k*-uniform hypergraph with edge density α is $\geq \omega\left(\alpha^{k+1}\right)$ and $\leq \alpha^{k+1-o(1)}$



a diamond

C.f. (Tao blog) Minimum diamond density in a graph with triangle-density β is $\geq \omega(\beta^2)$ and $\leq \beta^{2-o(1)}$

Higher dimensional corners?



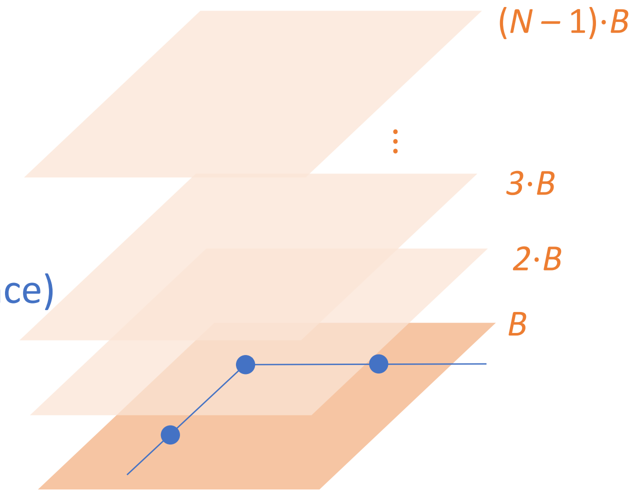
No popular difference for 3-dimensional corners!

Theorem (Fox—Sah—Sawhney—Stoner—Z.). $\forall\ 0 < \alpha < 1/2\ \ \exists\ A \subset [N]^3$:
$|A| \geq \alpha N^3$ but the number of 3-dim corners with common difference d
is $\leq \alpha^{c \log(1/\alpha)} N^3$ for every nonzero d

Stacking Behrend (simpler construction given in Sah—Sawhney—Z.)

Take a set $B \subset (\mathbb{Z}/N\mathbb{Z})^2$ with few corners, and layer dilates of B

density of 3-dim corners in the stack with fixed common difference
≤ density of 2-dim corners in the stack with a fixed common difference
= density of 2-dim corners in B (without restriction on common difference)



More general patterns: if a pattern contains has 3 points that do not affine span the whole space, then there is a construction with popular difference density ≤ $\alpha^{c \log(1/\alpha)}$

5-point patterns in 1-dim

Theorem (Bergelson–Host–Kra–Ruzsa '05). No popular common difference property for 5-AP

Extended to all 5-point patterns [Fox—Sah—Sawhney—Stoner—Z.]

Theorem. $\forall\ 0 < \alpha < 1/2\ \exists\ A \subset [N]$ with $|A| \geq \alpha N$ such that for every $d \neq 0$, the number of 5-APs in A with common difference d is $\leq \alpha^{c \log(1/\alpha)} N$

Recall Behrend constructed a 3-AP-free subset of $[L]$ of size $Le^{-O(\sqrt{\log L})}$, obtained as the image of a projection of a set of lattice points of some higher dimensional sphere

Why 5?

Related to: a quadratic curve intersects a sphere in ≤ 4 points

"Quadratic Behrend": there exists a subset of $[L]$ of size $Le^{-O(\sqrt{\log L})}$ that does not contain any pattern of the form $(P(0), P(1), P(2), P(3), P(4))$ for any nonconstant quadratic polynomial P

4-point patterns in 1-dim

Green–Tao's popular difference for 4-APs: quadratic Fourier analysis + "positivity": symmetric coefficients in

$$x^2 + 3(x + 2y)^2 = (x + 3y)^2 + 3(x + y)^2$$

Same argument apply to any 4-term pattern of the form $\{0, a, b, a + b\}$

What about other 4-point patterns? E.g., {0, 1, 2, 4}?

Question. Is it true that every $A \subset [N]$ with $|A| \geq \alpha N$ has some $y \neq 0$ such that

$$\#\{x : x, x + y, x + 2y, x + 4y \in A\} \geq \left(\alpha^4 - o(1)\right)N$$

Theorem (Sah–Sawhney–Z.). For every $0 < a_1 < a_2 < a_3$ with $a_3 \neq a_1 + a_2$, and $0 < \alpha < 1/2$, $\exists\, A \subset [N]$ with $|A| \geq \alpha N$ such that for all $y \neq 0$

$$\#\{x : x, x + a_1 y, x + a_2 y, x + a_3 y \in A\} \leq (1 - c)\alpha^4 N$$

where $c > 0$ is some absolute constant (and N assumed sufficiently large).

No popular difference for (0,1,2,5) pattern

Goal. Construction function $f\colon \mathbb{Z}/N\mathbb{Z} \to [0,1]$ with $\mathbb{E}f \geq \alpha$ and for every $y \neq 0$,

$$\mathbb{E}_x f(x)f(x+y)f(x+2y)f(x+5y) < 0.999\alpha^4$$

Construction. Let ω = N-th root of unity. Let

$$\frac{f(x)}{\alpha} = 1 + \gamma_1 \cdot (\omega^{6x^2} + \omega^{-6x^2}) + \gamma_2 \cdot (\omega^{15x^2} + \omega^{-15x^2}) + \gamma_3 \cdot (\omega^{10x^2} + \omega^{-10x^2}) + \gamma_4 \cdot (\omega^{x^2} + \omega^{-x^2})$$

After Gauss sum cancelations

$$\mathbb{E}f = \alpha + o(1)$$

and

$$\mathbb{E}_x f(x)f(x+y)f(x+2y)f(x+5y) = \alpha^4(1 + 2\gamma_1\gamma_2\gamma_3\gamma_4 + o(1))$$

Expectation
not taken over y

as the only possible nontrivial relation among the quadratic exponentials is

$$6x^2 - 15(x+y)^2 + 10(x+2y)^2 - (x+5y)^2 = 0$$

Conclude by picking a small negative γ_1 and small positive $\gamma_2, \gamma_3, \gamma_4$

No popular difference for (0,1,2,4) pattern

Goal. construction function $f\colon \mathbb{Z}/N\mathbb{Z} \to [0,1]$ with $\mathbb{E}f \geq \alpha$ and for every nonzero y,

$$\mathbb{E}_x f(x)f(x+y)f(x+2y)f(x+4y) < 0.999\alpha^4$$

Construction. Let ω = N-th root of unity. Let

$$\frac{f(x)}{\alpha} = 1 + \gamma_1 \cdot (\omega^{3x^2} + \omega^{-3x^2}) + \gamma_2 \cdot (\omega^{8x^2} + \omega^{-8x^2}) + \gamma_3 \cdot (\omega^{6x^2} + \omega^{-6x^2}) + \gamma_4 \cdot (\omega^{x^2} + \omega^{-x^2})$$

But now

$$\mathbb{E}_x f(x)f(x+y)f(x+2y)f(x+5y)$$
$$= \alpha^4\left(1 + 2\gamma_1\gamma_2\gamma_3\gamma_4 + \gamma_1^2\gamma_3^2\left(\omega^{30y^2} + \omega^{-30y^2}\right) + \gamma_1^2\gamma_3(\omega^{24y^2} + \omega^{6y^2} + \omega^{-6y^2} + \omega^{-24y^2}) + o(1)\right)$$

Expectation
not taken over *y*

Extra terms due to additional linear relations among the quadratic exponentials, e.g.,

$$6x^2 - 6(x+y)^2 - 3(x+2y)^2 + 3(x+4y)^2 = 30y^2$$

Fortunately, in this case, we can still conclude by picking a small negative γ_1 and small positive $\gamma_2, \gamma_3, \gamma_4$

In general, rule out/handle all such linear relations (computer assisted; rational points on hyperelliptic curves)

4-point patterns in 1-dim

Theorem (Sah–Sawhney–Z.). For every $0 < a_1 < a_2 < a_3$ with $a_3 \neq a_1 + a_2$, and $0 < \alpha < 1/2$,
$\exists$ $A \subset [N]$ with $|A| \geq \alpha N$ such that for all $y \neq 0$

$$\#\{x : x, x + a_1 y, x + a_2 y, x + a_3 y \in A\} \leq (1 - c)\alpha^4 N$$

where $c > 0$ is some absolute constant (and N assumed sufficiently large).

Also for certain patterns we can do much better:

for every $C > 0$, there exist 4-pt patterns where the RHS can be replaced by $< \alpha^C N$

Related to open problem of the type: max size of subset of $[N]$
avoiding $x + 3y = 2z + 2w$ (known: between $\sqrt{N}$ and $o(N)$)

No Behrend–type constructions

A question of Gowers

Basic idea of Fourier analytic proof of Roth's theorem: a Fourier uniform subset of $\mathbb{Z}/N\mathbb{Z}$ with density α has 3-AP density $= \alpha^3 - o(1)$

False for 4-APs

The standard example $\{x : x^2 \bmod N < \alpha N\}$ has *more* 4-APs than expected ($\approx C\alpha^3 \geq \alpha^4$),
so not an obstruction to the density increment argument

Question. Can a Fourier uniform set have 4-AP density much less than α^4?

Gowers constructed a example with 4-AP density $< \alpha^{4+c}$

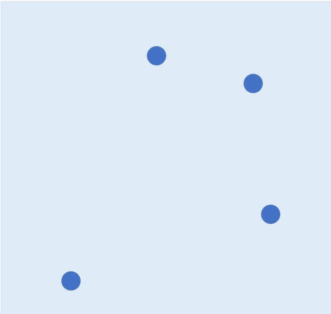
Question (Gowers). Must a Fourier-uniform set of density α have 4-AP density $\geq \alpha^{1000}$?

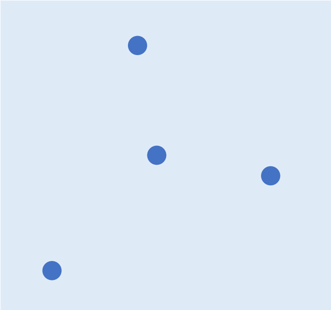
We don't know. But the result on the previous slide implies that "4-AP" cannot be replaced by an "arbitrary 4-point pattern" (if one is not allowed to change the "1000")

4-point patterns: 2-dim

Theorem (Sah–Sawhney–Z.). For every 4-point pattern P in $\mathbb{Z}^2$, and fixed $0 < \alpha < 1/2$, $\exists\, A \subset [N]^2$ so that $\forall\, y \neq 0$, the number of translates of $y \cdot P$ in A is $\leq \alpha^5 e^{C\sqrt{\log(1/\alpha)}} N^3$

For every nonconvex 4-point pattern, result can be improved to $\leq \alpha^{c \log(1/\alpha)} N^3$





Roth's theorem with popular difference (Green '05). $\forall\, \epsilon > 0\ \exists\, N_0(\epsilon)$ so that for all $N \geq N_0(\epsilon)$ and $A \subset [N]$ with $|A| \geq \alpha N$, $\exists\, d \neq 0$ such that the number of 3-APs in A with common difference d is $\geq (\alpha^3 - \epsilon)N^2$

Theorem (Fox—Pham—Z.). The optimal $N_0(\epsilon)$ is $\mathrm{tower}\big(\Theta(\log 1/\epsilon)\big)$

$P \subseteq \mathbb{Z}^r$	Popular difference density	
3 points in $\mathbb{Z}$	$\mathsf{pdd}_P(\alpha) = \alpha^3$	[Green '05]
$k_1 < k_2 < k_3 < k_4$ in $\mathbb{Z}$ with $k_1 + k_4 = k_2 + k_3$	$\mathsf{pdd}_P(\alpha) = \alpha^4$	[Green–Tao '10]
Other 4 point patterns in $\mathbb{Z}$	$\mathsf{pdd}_P(\alpha) < (1-c)\alpha^4$	[Sah–Sawhney–Z.]
Affine dim of $P < r$	$\mathsf{pdd}_P(\alpha) < \alpha^{c \log(1/\alpha)}$	[Sah–Sawhney–Z.]
3 non-collinear points in $\mathbb{Z}^2$	$\omega(\alpha^4) \leq \mathsf{pdd}_P(\alpha) \leq \alpha^{4-o(1)}$	[Mandache/Berger/ Fox—Sah—Sawhney—Stoner—Z.]
4 points in strict convex position in $\mathbb{Z}^2$	$\mathsf{pdd}_P(\alpha) < \alpha^{5-o(1)}$	[Sah–Sawhney–Z.]
4 points in nonconvex position in $\mathbb{Z}^2$	$\mathsf{pdd}_P(\alpha) < \alpha^{c \log(1/\alpha)}$	[Sah–Sawhney–Z.]
At least 5 points	$\mathsf{pdd}_P(\alpha) < \alpha^{c \log(1/\alpha)}$	[Bergelson–Host–Kra–Ruzsa / Fox—Sah—Sawhney—Stoner—Z.]
Affine dimension at least 3	$\mathsf{pdd}_P(\alpha) < \alpha^{c \log(1/\alpha)}$	[Fox—Sah—Sawhney—Stoner—Z.]