THE JOINTS PROBLEM FOR VARIETIES

arXiv:2008.01610

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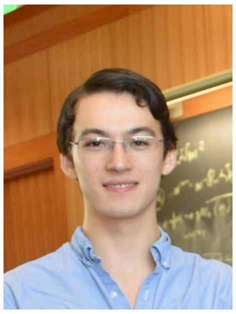
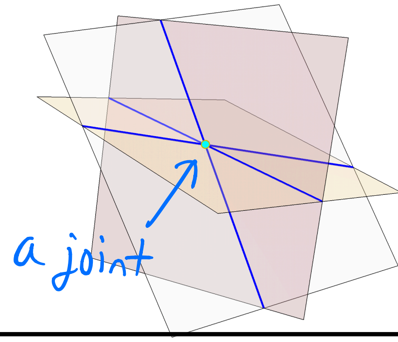
with



Jonathan Tidor & Hung-Hsun Hans Yu

Joints problem What's the max # of joints that N lines in $\mathbb{R}^3$ can make?

A **joint** is a point contained in 3 non-coplanar lines

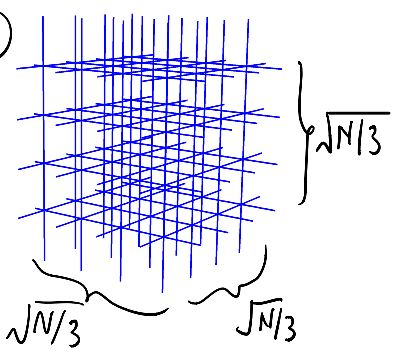


Examples

N lines

$\Theta(N^{3/2})$ joints

①

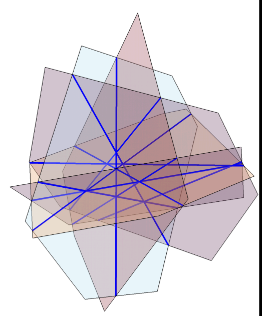


$\left(\frac{N}{3}\right)^{3/2}$ joints

② $k \sim \sqrt{2N}$ generic planes

$\rightsquigarrow$ pairwise form $\binom{k}{2} \sim N$ lines

& triplewise form $\binom{k}{3} \sim \frac{\sqrt{2}}{3} N^{3/2}$ joints



Introduced by Chazelle–Edelsbrunner–Guibas–Pollack–Seidel–Sharir–Snoeyink '92 $O(N^{7/4})$

Guth–Katz (2010) : N lines in $\mathbb{R}^3$ form $O(N^{3/2})$ joints

Subseq. generalized to arb dim & fields ($\mathbb{F}^d$) Kaplan–Sharir–Shustin, Quilodrán

Yu–Z. (2019+) : optimal const, $\leq \frac{\sqrt{2}}{3} N^{3/2}$ joints

Connections

- Kakeya problem (Wolff)
- Finite field Kakeya problem (Dvir) ← polynomial method
- Multilinear Kakeya, "joints of tubes" (Bennett–Carbery–Tao, Guth)

Joints of flats: max # joints for N planes in $\mathbb{F}^6$?

a point contained in a triple of planes in spanning & indep directions

2-dim flats

Construction $\Theta(N^{3/2})$ joints: generic 4-flats

pairwise intersect → planes

triplewise intersect → joints

Why I like this problem:

- natural extension of the joints problem
- a key step in pf of joints thm fails badly
- need a new extension of the polynomial method

Incidence geometry for higher dimensional objects

Prior results on joints of higher dim objects

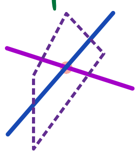
[Yang] N planes in $\mathbb{R}^6$ have $N^{\frac{3}{2}+o(1)}$ joints

Techniques: Guth–Katz — polynomial partitioning
Guth — partitioning varieties
Solymosi–Tao — bounded degree

Limitations
1. Error term
2. Only $\mathbb{R}$

[Yu–Z. / Carbery–Iliopoulou] N lines & M planes in $\mathbb{F}^4$ make $O(NM^{1/2})$ joints

(plane–line^2)



line–line–plane, in indep & spanning directions

Our results [Tidor–Yu–Z.]

$\boxed{\text{Joints of flats} \quad N \text{ planes in } \mathbb{F}^6 \text{ have } O(N^{3/2}) \text{ joints}}$

$\boxed{\text{Joints of varieties} \quad \text{A set of 2-dim varieties in } \mathbb{F}^6 \text{ of total degree } N \text{ has } O(N^{3/2}) \text{ joints}}$

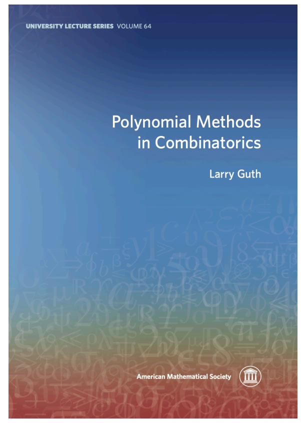
$p \in V_1, V_2, V_3$ regular point
tangent planes at p spanning & indep directions

And more generally:

- arbitrary dimensions
- several sets of varieties (multijoints)
- counting joints with multiplicities

prev. for joints of lines
- conj. [Carbery]
- [Iliopoulou]
- [Zhang]

Review of the proof of: [Kaplan–Sharir–Shustin, Quilodrán]



Thm N lines in $\mathbb{R}^3$ have $O(N^{3/2})$ joints

① Parameter counting:

using $\dim \mathbb{R}[x_1, \dots, x_d]_{\le n} = \binom{n+d}{d}$ (deg $\le n$)

deduce that $\exists$ non-zero poly g, $\deg \le C J^{1/3}$, vanishing on joints

Take g with min deg. $J = \#$joints

② Vanishing lemma: a single-variable polynomial cannot vanish more times than its degree

③ A joints-specific argument. If all lines have $> C J^{1/3}$ joints, then vanishing lemma $\Rightarrow$ g vanishes on all lines $\Rightarrow$ ∇g vanishes on all joints

$\Rightarrow$ one of $\partial_x g, \partial_y g, \partial_z g$ is nonzero, lower deg & vanish on all joints

So some line has $\le C J^{1/3}$ joints. Remove this line & repeat

$J \le C J^{1/3} N$ Thus $J = O(N^{3/2})$

How to generalize vanishing lemma to 2-var polynomials?

$\boxed{\text{Thm (Tidor–Yu–Z.) } N \text{ planes in } \mathbb{R}^6 \text{ have } O(N^{3/2}) \text{ joints}}$

Above proof would generalize if...

Attempt I

$g \in \mathbb{R}[x,y]_{\le n}$ vanishing at $\binom{n+2}{2}$ distinct points $\overset{???}{\Longrightarrow}$ $g \equiv 0$

$\boxed{\text{NO}}$

Method of multiplicities:

Attempt II

$g \in \mathbb{R}[x,y]_{\le n}$ vanishing to order $> n$ at a single point $\overset{???}{\Longrightarrow}$ $g \equiv 0$

i.e. $\frac{\partial^{i+j} g}{\partial x^i \partial y^j}(p) = 0 \quad \forall\, i+j \le n$

$\boxed{\text{YES, but how does it help?}}$

Attempt III

$\underbrace{g \in \mathbb{R}[x,y]_{\le n}}_{\sim \frac{n^2}{2} \text{ dim}}$ $\underbrace{\text{vanishing to order } s \text{ at } \approx \frac{n^2}{s^2} \text{ points}}_{\sim \frac{n^2}{2} \text{ linear constraints}}$ $\overset{???}{\Longrightarrow}$ $g \equiv 0$

$\boxed{\text{NO}}$ e.g. $g(x,y) = y^s$

Linear dependencies among vanishing conditions

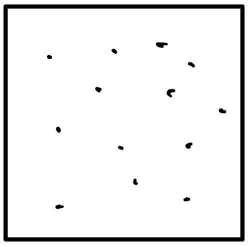
linear constraints on $g \in \mathbb{R}[x,y]_{\le n}$. e.g. $(\partial_{xx} - \partial_{yy})g(p) = 0$ for some fixed p

- viewed as both (derivative op, point)

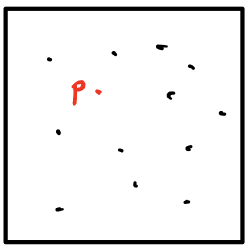
  & linear functionals on $\mathbb{R}[x,y]_{\le n}$

★ Key idea 1 Collecting linearly indep vanishing conditions

Restricting to a plane for now



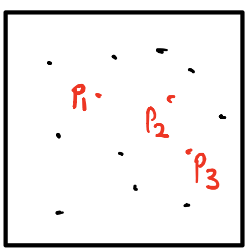
We will construct a set of $\dim \mathbb{R}[x,y]_{\leq n} = \binom{n+2}{2}$ linearly indep vanishing conditions on $\mathbb{R}[x,y]_{\leq n}$



Attached to each point p is a set of vanishing conditions for $g \in \mathbb{R}[x,y]_{\leq n}$:

$g(p)=0$, $\partial_x g(p)=0$, $\partial_y g(p)=0$

$\partial_{xx} g(p)=0$, $\partial_{xy} g(p)=0$, $\partial_{yy} g(p)=0$, $\partial_{xxx} g(p)=0, \ldots$



The above vanishing conditions attached to several different points are lin. dep. as linear functionals on $\mathbb{R}[x,y]_{\leq n}$

We will select a basis of linear functionals on $\mathbb{R}[x,y]_{\leq n}$ via the following procedure.

First attempt

Cycle through the points on the plane

$p_1, p_2, p_3, \ldots, p_1, p_2, p_3, \ldots, p_1, p_2, p_3, \ldots$

p_1 : add vanishing condition $g(p_1)=0$

p_2 : add vanishing condition $g(p_2)=0$ if nonredundant

$\vdots$

p_1 : add a nonredundant subset of $\partial_x g(p_1)=0$, $\partial_y g(p_1)=0$

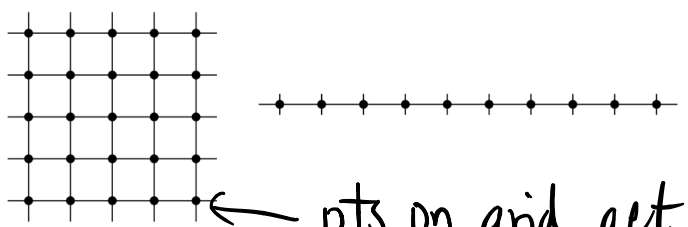
none implied by other added + prev. added
i.e. basis extension

p_2 : add a nonredundant subset of $\partial_x g(p_2)=0$, $\partial_y g(p_2)=0$

and so on...

Can we control the # van. cond. attached to each pt?

Example



pts on grid get way more vanishing conditions than pts on the line

UNDESIRABLE

(This example also comes up for inverse Bézout; see Tao blog)

☆ Key idea 2 Let some points get a head start

e.g. (100 pts) $p_1, p_2, \dots, p_{50}, \; p_1, \dots, p_{50}, \; \dots, \; p_1, \dots, p_{50}, \; p_1, \dots, p_{100}, \; p_1, \dots, p_{100}, \; \dots$

Handicap $\vec{\alpha} \in \mathbb{Z}^J$ assigns an integer to each point

e.g.

points	a	b	c	d	e
handicap	0	1	3	0	-1

→ order: c c b c a b c d a b c d e a b c d e ...

Modify process of assigning vanishing conditions

c : add a nonredundant set of 0th order derivative vanishing @ c

c ———— 1st ———— c

b ———— 0th ———— b

c ———— 2nd ———— c

a ———— 0th ———— a

Want a "good" choice of handicaps: treating all joints "fairly"

$\vec{\alpha} \in \mathbb{Z}^J$

Handicap $\longmapsto$ partition of $\binom{n+2}{2}$ among joints

$\binom{n+2}{2} = \dim \mathbb{R}[x,y]_{\le n}$ (# vanish. cond. assigned)

Hard to compute!

☆ Key idea 3: Existence of good handicap via compactness/smoothing

① Monotonicity $\quad \alpha_p \nearrow \;\Rightarrow$ # van. cond at p cannot $\searrow$

② Lipschitz continuity $\quad$ small Δ in handicap $\rightsquigarrow$ small Δ in # van cond

③ Bounded domain $\quad$ suffices to consider handicaps with bounded values (else some pt gets no van. cond.)

Putting different planes together

Handicap $\vec{\alpha} \in \mathbb{Z}^J$ assigns an integer to each joint

Separately for each plane F, apply above process to assign vanishing conditions restricted to F to joints on F ← (derivative op, point)

A new vanishing lemma Given $0 \neq g \in \mathbb{R}[x_1, \dots, x_6]_{\leq n}$, $\exists$ joint p, contained in planes F_1, F_2, F_3 (indep & spanning directions) & derivative operator D_1 assigned to p on F_1 (& likewise D_2, D_3) s.t. $D_1 D_2 D_3 g(p) \neq 0$.

Remark (a) We are assigning only a small # possible (D,p), else claim is trivial

(b) The proof relies on (D,p)'s coming from the procedure earlier

By parameter counting,

linear constraints

$$\sum_{\text{joints } p} \underbrace{(\#\, D_1 @ p \text{ on } F_1)(\#\, D_2 @ p \text{ on } F_2)(\#\, D_3 @ p \text{ on } F_3)}_{f(\vec{\alpha}, p)} \geq \dim \mathbb{R}[x_1, \dots, x_6]_{\leq n} = \binom{n+6}{6}$$

By compactness/smoothing, considering the handicap $\vec{\alpha}$ that minimizes

$$\max_p f(\vec{\alpha}, p) - \min_p f(\vec{\alpha}, p)$$

we deduce that $\exists \vec{\alpha}$ s.t. $\text{''} = o(n^6)$

Total $\binom{n+2}{2}$ vanishing cond. assigned to each plane

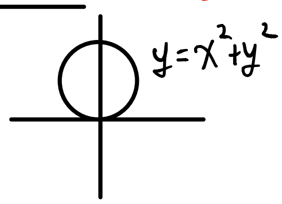
Putting together + AM-GM $\Rightarrow$ joints of flats theorem $\square$

Joints of varieties

Flats: higher order directional directives along a flat

Varieties: derivatives in local coordinates

e.g.



On the circle,

$$
\begin{aligned}
y &= x^2+y^2 \\
&= x^2+(x^2+y^2)^2 \\
&= x^2+(x^2+(x^2+y^2)^2)^2 = \cdots \\
&= x^2+x^4+2x^6+\cdots
\end{aligned}
$$

Power series in local coord x

$\boxed{\text{Completion}}$

2^{nd} order derivative operator at the origin is $\frac{1}{2}\frac{\partial^2}{\partial x^2}+\frac{\partial}{\partial y}$ (not $\frac{\partial^2}{\partial x^2}$) so that evaluations give linear functional on the space of regular functions

Extension to arbitrary fields $\mathbb{F}$

When differentiating, we only care about coeff extraction

Hasse derivatives (formal algebraic derivatives)

Question Other applications of this variant of polynomial method for higher dim objects?