Independent Sets and Graph Homomorphisms

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Abstract: We prove a conjecture of Alon on the number of independent sets in a regular graph, and then generalize to graph homomorphisms.

Independent Sets

Let $G = (V, E)$ be a graph. An **independent set** is a subset of the vertices with no two adjacent.

Question. In the family of N-vertex, d-regular graphs G, when is the number of independent sets maximized?

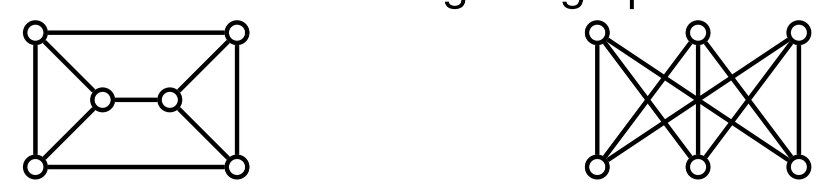
Theorem 1 (Zhao [4]; conjectured by Alon [1] in 1991 and proved for G bipartite by Kahn [3] in 2001)**.** For any N-vertex, d-regular graph G,

$$i(G) \le i\,(K_{d,d})^{N/2d} = \left(2^{d+1} - 1\right)^{N/(2d)},$$

where $i(G)$ denotes the number of independent sets of G. Note equality holds if G is a disjoint union of $K_{d,d}$'s.

All results have weighted generalizations.

Example: Two 6-vertex 3-regular graphs:

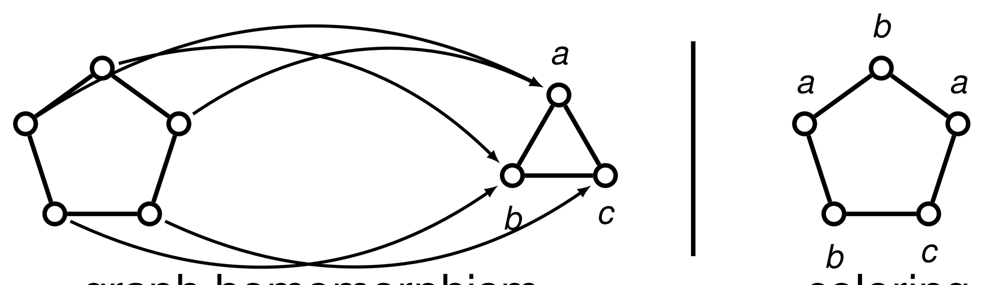


13 independent sets | 15 independent sets

Graph Homomorphisms

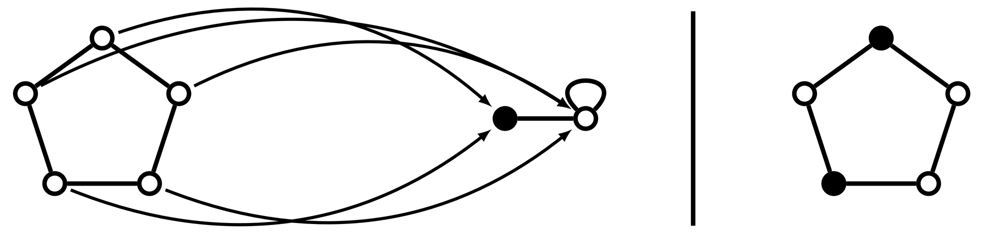
For graphs G and H (allowing loops for H), a **graph homomorphism** from G to H is a map from $V(G)$ to $V(H)$ so that all every edge of G gets carried to some edge of H. Denote the set of graph homomoprhisms from G to H by $\mathrm{Hom}(G, H)$.

Graph homomorphisms generalize graph colorings by choosing $H = K_k$ for a k-coloring.



graph homomorphism ⟷ coloring

Graph homomorphisms also generalize the independent sets, by choosing $H =$.



graph homomorphism ⟷ independent set

Goal. Generalize Thm 1 to graph homomorphisms.

Question. For a fixed graph H (allowing loops), in the family of N-vertex, d-regular simple graphs G, when is $|\mathrm{Hom}(G, H)|$ maximized?

It is known that Theorem 1 generalizes at least in the bipartite case.

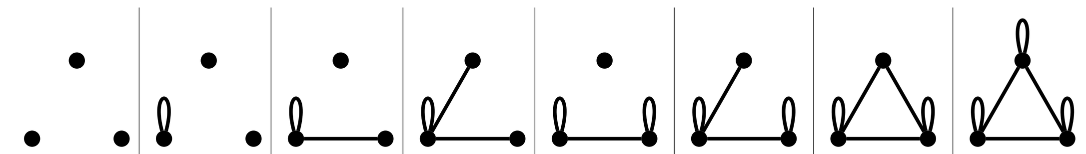
Theorem 2 (Galvin & Tetali [2])**.** For any H (allowing loops), and any N-vertex, d-regular bipartite graph G,

$$|\mathrm{Hom}(G, H)| \le |\mathrm{Hom}(K_{d,d}, H)|^{N/(2d)}.$$

Theorem 3 ([5])**.** The bipartite condition on G in Theorem 2 can be dropped if H is a *bipartite swapping target*.

Bipartite Swapping Targets

- The family of bipartite swapping targets is defined by some technical conditions that we omit here.
- H is a bipartite swapping target if and only if some auxilliary graph is bipartite, so the property is easy to check.
- **An easy-to-describe subclass:** Any graph whose adjacency matrix can be written so that the 1's form a Young diagram is a bipartite swapping target. E.g., all such 3-vertex graphs:

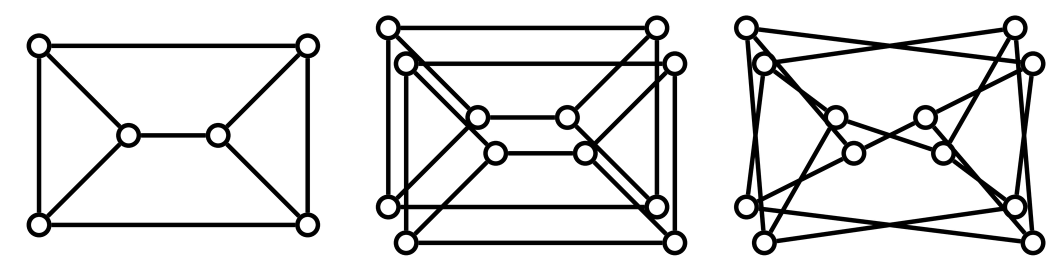


0 0 0	1 0 0	1 1 0	1 1 1	1 1 0	1 1 1	1 1 1	1 1 1
0 0 0	0 0 0	1 0 0	1 0 0	1 1 0	1 1 0	1 1 1	1 1 1
0 0 0	0 0 0	0 0 0	1 0 0	0 0 0	1 0 0	1 1 0	1 1 1

Bipartite Swapping Trick

Idea: Reduce general graph G to bipartite graph by comparing $G \sqcup G$ with $G \times K_2$ and finding an injection

$$\mathrm{Hom}(G \sqcup G, H) \longrightarrow \mathrm{Hom}(G \times K_2, H)$$

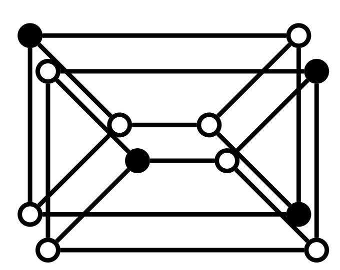


G | $G \sqcup G$ | $G \times K_2$

Suppose that we have such an injection, since $G \times K_2$ is bipartite, Theorem 2 would imply that

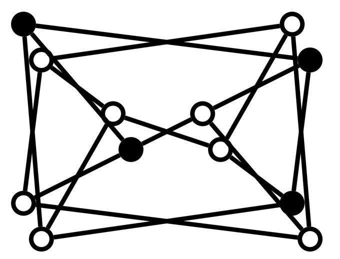
$$|\mathrm{Hom}(G, H)| = |\mathrm{Hom}(G \sqcup G, H)|^{1/2} \le |\mathrm{Hom}(G \times K_2, H)|^{1/2} \le |\mathit{Hom}(K_{d,d}, H)|^{N/(2d)}.$$

Construction of injection: (for independent sets)



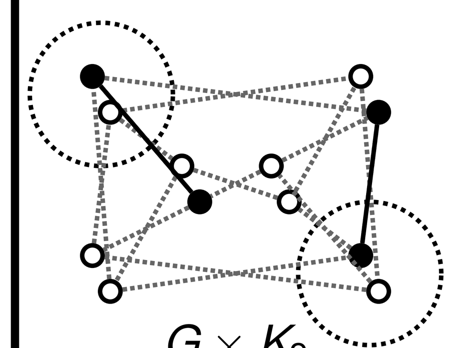
$G \sqcup G$

Start with an independet set (black vertices) of $G \sqcup G$



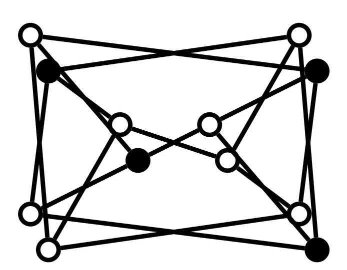
$G \times K_2$

Cross edges to get $G \times K_2$. However, the same subset of vertices might no longer be an independent set.



$G \times K_2$

The set of "violated edges" turns out to form a bipartite graph. Select the lexigraphically first set of vertex-pairs that gives a bipartition of the violated edges.



$G \times K_2$

Swap the vertices in the pairs selected in the previous step. This gives us an independent set of $G \times K_2$.

Remark: Bipartite swapping property of H allows this technique to be generalized to $\mathrm{Hom}(G, H)$.

Stable Set Polytope

The **stable set polytope** of G is defined as the convex hull in $\mathbb{R}^{V(G)}$ of the characteristic vectors of the independent sets of G.

Theorem 4 ([5])**.** For any N-vertex, d-regular graph G, we have

$$i_V(G) \le i_V(K_{d,d})^{N/(2d)} = \binom{2d}{d}^{-N/(2d)},$$

where $i_V(\cdot)$ is the volume of the stable set polytope.

Graph Colorings

Unfortunately, K_d does not pass our test for a bipartite swapping target, so our technique cannot solve the graph coloring case. However, using the bipartite swapping trick, we can still prove the following result.

Theorem 5 ([5])**.** For every N-vertex, d-regular graph G,

$$|\mathrm{Hom}(G, K_q)| \le |\mathrm{Hom}(K_{d,d}, K_q)|^{N/(2d)} \qquad (1)$$

for all sufficiently large q. Note that $|\mathrm{Hom}(-, K_q)|$ counts the number of proper q-colorings.

Conjecture 6. The inequality (1) holds for all q.

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References

[1] N. Alon, *Independent sets in regular graphs and sum-free subsets of finite groups*, Israel J. Math. **73** (1991), no. 2, 247–256.

[2] D. Galvin and P. Tetali, *On weighted graph homomorphisms*, Graphs, morphisms and statistical physics, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., vol. 63, Amer. Math. Soc., Providence, RI, 2004, pp. 97–104.

[3] J. Kahn, *An entropy approach to the hard-core model on bipartite graphs*, Combin. Probab. Comput. **10** (2001), no. 3, 219–237.

[4] Y. Zhao, *The number of independent sets in a regular graph*, Combin. Probab. Comput. **19** (2010), no. 2, 315–320.

[5] ______, *The bipartite swapping trick for graph homomorphisms*, SIAM J. Discrete Math. (to appear).