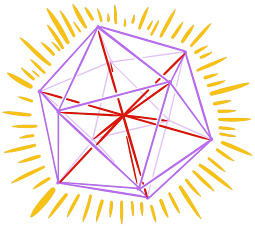
# Extremal problems in discrete geometry

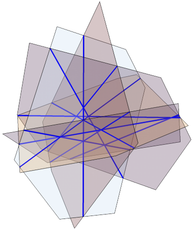
Yufei Zhao

MIT

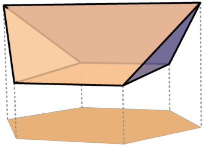
CanaDAM 2021



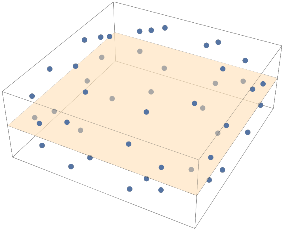
Equiangular lines



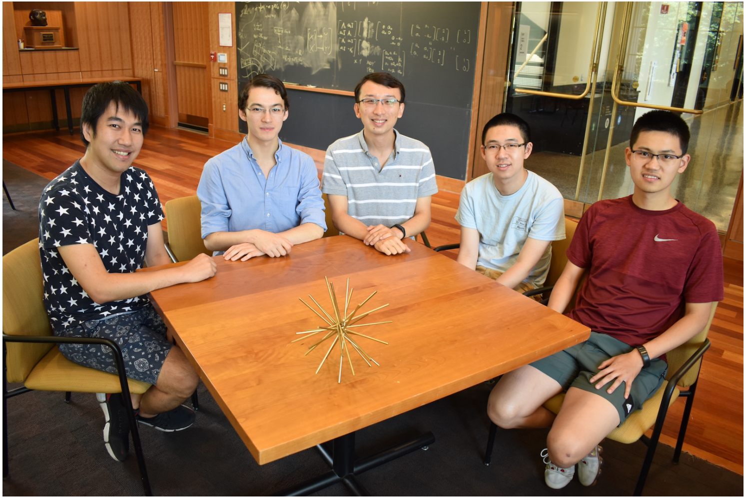
Joints



Extension complexity



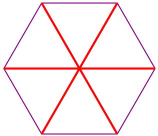
Width of transitive sets



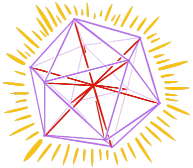
Zilin Jiang, Jonathan Tidor, me, Yuan Yao, Shengtong Zhang

# Equiangular lines

$$N(d) = \max \# \text{ of lines in } \mathbb{R}^d \text{ with pairwise equal angles}$$



$N(2) = 3$



$N(3) = 6$

Exact answer known for finitely many $d$.

[de Caen '00] $$cd^2 \leq N(d) \leq \binom{d+1}{2}$$ [Gerzon '73]

In the lower bound construction, pairwise angles $\to 90^\circ$ as $d \to \infty$.

# Equiangular lines with a fixed angle

$$N_\alpha(d) = \max \# \text{ of equiangular lines in } \mathbb{R}^d \text{ with pairwise angles } \cos^{-1}\alpha$$

$(\alpha > 0$ fixed, $d \to \infty)$

**History:**

- [Lemmens, Seidel '73] $N_{1/3}(d) = 2(d-1)$ for all $d \geq 15$
- [Neumaier '89] $N_{1/5}(d) = \left\lfloor \frac{3}{2}(d-1) \right\rfloor$ for all sufficiently large $d$

  *"the next interesting case, [$N_{1/7}(d)$], will require substantially stronger techniques"*

- [Bukh '16] $N_\alpha(d) \leq C_\alpha d$
- [Balla, Dräxler, Keevash, Sudakov '18] $\forall \alpha \neq \frac{1}{3}$, $N_\alpha(d) \leq (1.93 + o(1))d$
- [Jiang, Polyanskii '20] $\forall \alpha \notin \{\frac{1}{3}, \frac{1}{5}, \frac{1}{1+2\sqrt{2}}\}$. $N_\alpha(d) \leq (1.49 + o(1))d$

**Problem.** Determine, for each fixed $\alpha$, $\displaystyle\lim_{d\to\infty} \frac{N_\alpha(d)}{d}$

## Our work completely solves this problem

**Theorem (Jiang, Tidor, Yao, Zhang, Z., '19+)**

For every integer $k \geq 2$,

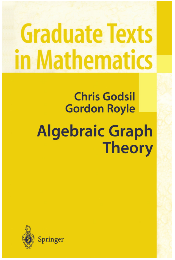
$$N_{\frac{1}{2k-1}}(d) = \left\lfloor \frac{k}{k-1}(d-1) \right\rfloor \qquad \forall d \geq d_0(k).$$

Furthermore, for every fixed $\alpha \in (0,1)$, setting $\lambda = \dfrac{1-\alpha}{2\alpha}$ and "spectral radius order"

$$k = k(\lambda) = \min\{|V(G)| : \text{ graph } G \text{ whose adjacency matrix has top eigenvalue } \lambda\}$$

($k(\lambda) = \infty$ if no such $G$ exists). We have

$$N_\alpha(d) = \begin{cases} \left\lfloor \frac{k}{k-1}(d-1) \right\rfloor & \forall d \geq d_0(\alpha) \quad \text{if } k < \infty \\ d + o(d) & \text{if } k = \infty \end{cases}$$




Equiangular lines: *"one of the founding problems of algebraic graph theory"*

## Equiangular lines $\longleftrightarrow$ graphs

- Given unit vectors $v_1, \ldots, v_N \in \mathbb{R}^d$, with $\langle v_i, v_j \rangle = \pm\alpha$ for all $i \neq j$
- Associate a graph on vertex set $[N]$, with $i \sim j$ if $\langle v_i, v_j \rangle = -\alpha$

Key ingredient in our proof: a new result in spectral graph theory

**Theorem (Jiang, Tidor, Yao, Zhang, Z., '19+)**

A connected $n$-vertex graph with maximum degree $\leq \Delta$ has second eigenvalue multiplicity $O_\Delta\left(\dfrac{n}{\log \log n}\right)$.

# Spherical sets with two fixed distances

Fix $-1 < \beta < 0 < \alpha < 1$. (Equiangular lines: $-\alpha = \beta$)

$N_{\alpha,\beta}(d) = \max \#$ unit vectors in $\mathbb{R}^d$ whose pairwise inner products lie in $\{\alpha, \beta\}$

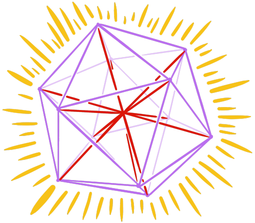
[Bukh '16] $N_{\alpha,\beta}(d) = O_{\alpha,\beta}(d)$

**Problem.** Determine $\lim_{d\to\infty} \frac{N_{\alpha,\beta}(d)}{d}$.

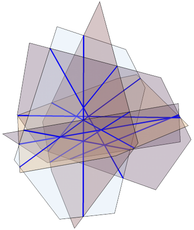
In our follow-up paper [Jiang, Tidor, Yao, Zhang, Z., '20+], we

- Conjectured the limit in terms of eigenvalue multiplicities of certain **signed graphs**
- Proposed a framework towards proving this conjecture
- Solved the problem when

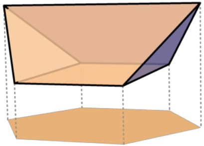
$$\alpha + 2\beta < 0 \qquad \text{or} \qquad \frac{1-\alpha}{\alpha-\beta} \in \{1, \sqrt{2}, \sqrt{3}\}.$$



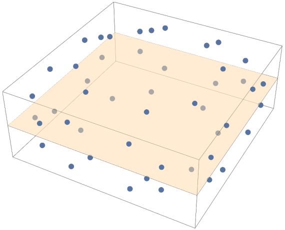
Equiangular lines



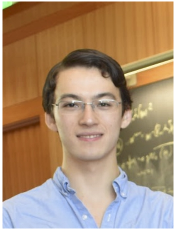
Joints



Extension complexity



Width of transitive sets



Jonathan Tidor

Hung-Hsun Hans Yu

# The joints problem

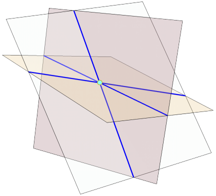
[Chazelle, Edelsbrunner, Guibas, Pollack, Seidel, Sharir, Snoeyink '92]

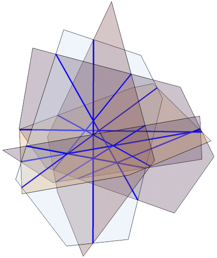
**Question.** Max # of joints formed by $N$ lines in $\mathbb{R}^3$?

A joint is a point contained in 3 non-coplanar lines

**Example.**

- Start with $k$ generic planes in $\mathbb{R}^3$
- pairwise intersections: $\binom{k}{2} = N$ lines
- triplewise intersections: $\displaystyle\binom{k}{3} \sim \frac{\sqrt{2}}{3} N^{3/2}$ joints





The joints problem is connected to the Kakeya problem

**Joints theorem.** (Guth, Katz, '10)

$N$ lines in $\mathbb{R}^3$ form $O(N^{3/2})$ joints

- Polynomial method
- True for arbitrary dimension and fields ($\mathbb{F}^d$) (Kaplan–Sharir–Shustin, Quilodrán)

**Theorem — optimal constant in joints theorem** (Yu, Z., '19+)

$N$ lines in $\mathbb{R}^3$ form $\leq \frac{\sqrt{2}}{3} N^{3/2}$ joints

- Also extends to all dimensions and fields

# Joints of flats

**Question.** Max # joints formed by $N$ planes in $\mathbb{F}^6$?

"joint" = a point contained in 3 planes with independent directions

**Example.** $\Theta(N^{3/2})$ joints. Start with some generic 4-flats, pairwise intersections give planes, and triplewise intersections give joints.

**Theorem.** (Yang '16) $N$ planes in $\mathbb{R}^6$ form $N^{3/2+o(1)}$ joints

- Only works over $\mathbb{R}$, and has $+o(1)$ error term in exponent
- Technique: polynomial partitioning using bounded degree polynomials

**Theorem — Joints of planes (Tidor, Yu, Z. '20+)**

$N$ planes in $\mathbb{F}^6$ have $O(N^{3/2})$ joints

- Extends to every dimension

### Theorem — Joints of varieties (Tidor, Yu, Z. '20+)

A set of 2-dimensional varieties in $\mathbb{F}^6$ of total degree $N$ has $O(N^{3/2})$ joints

"joint" = a point contained as a regular point in three varieties with tangent planes in independent directions

More generally:

- Arbitrary dimensions
- Several sets of varieties ("multijoints")
- Counting joints with multiplicities

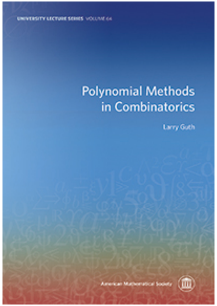
# Polynomial method

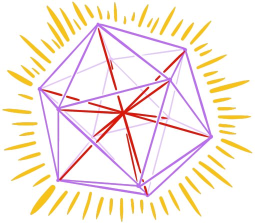
[Dvir '09] Solution to the finite field Kakeya problem

Two important ingredients

- **Parameter counting:** deducing the existence of a non-zero multivariate polynomial of small degree that has prescribed vanishings
- **Vanishing lemma:** If a degree $D$ polynomial vanishes at $> D$ points on a line, then it vanishes on the entire line.

Key difficulty for joints of planes: extending the vanishing lemma to planes

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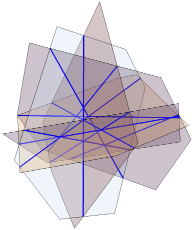
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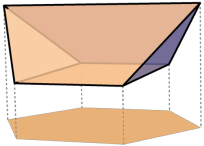
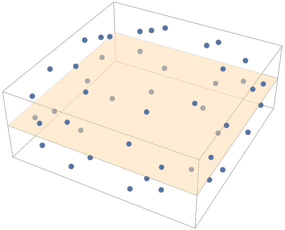
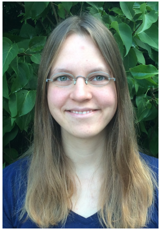

Extension complexity


Width of transitive sets

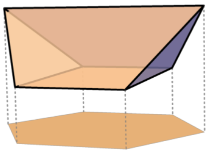
Matt Kwan



Lisa Sauermann

# Extension complexity of polytopes

**Definition.** The extension complexity of a polytope $P$ is the minimum number of facets in a (higher dimensional) polytope $Q$ such that one can write $P$ as the image of $Q$ under a linear projection.



A 3-dim polytope with 5 facets whose projection is a regular hexagon.

| **Polytopes** | | **Linear programming** |
|---|---|---|
| facets | $\longleftrightarrow$ | inequalities |
| new dimensions | $\longleftrightarrow$ | new auxiliary variables |

**Question.** Can every low-dimensional polytope be represented as the projection of some high-dimensional polytope with few facets?

# Extension complexity of polygons

**Question.** Maximum extension complexity of an $n$-gon?

- For a long time, it was believed that the answer is $n$.
- [Shitov '14] [Padrol, Pfeifle '15] $\leq 6n/7$
- [Shitov '14] $o(n)$
- [Shitov '20] $O(n^{2/3})$
- [Fiorini, Rothvoss, Tiwary 2012] A generic $n$-gon has extension complexity $\geq c\sqrt{n}$

**Open question.** Does every $n$-gon have extension complexity $O(\sqrt{n})$?

**Theorem (Kwan, Sauermann, Z., '20+)**

Every **cyclic** $n$-gon has extension complexity $O(\sqrt{n})$.

# Extension complexity of low dimensional polytopes

**Question.** Max extn. compl. of $d$-dim polytope with $n$ vertices (or, equivalently, $n$ facets)? (Especially for fixed $d$ and large $n$)

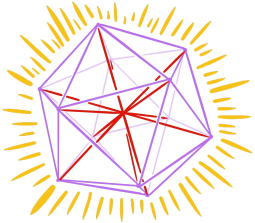
- [Padrol '16] Generic $d$-dimensional $n$-vertex polytope has extn. compl. $\geq c\sqrt{dn}$.
- No non-trivial upper bounds known

**Theorem.** (Kwan, Sauermann, Z. '20+)
"**random** polytopes in fixed dim. have low extension complexity"

- With probability $1 - o(1)$, the polytope with vertices $n$ uniform random points on a sphere in $\mathbb{R}^d$ has extension complexity $\Theta_d(\sqrt{n})$
- With probability $1 - o(1)$, the convex hull of $m$ uniform random points in a ball in $\mathbb{R}^d$ has extension complexity $\Theta_d(\sqrt{n})$ where $n = m^{(d-1)/(d+1)}$
(the typical number of vertices and facets is known to be $\Theta(n)$)

**Theorem.** (Kwan, Sauermann, Z. '20+)
"construction of low dim. polytope with high extension complexity"

- $\forall n \; \exists$ $n$-vertex polytope in dim. $\leq n^{o(1)}$ with extn. compl. $\geq n^{1-o(1)}$.



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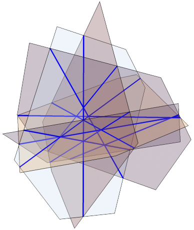

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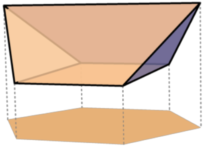

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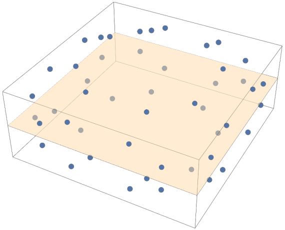


Width of transitive sets

Ashwin Sah

Mehtaab Sawhney

# Geometry of transitive sets

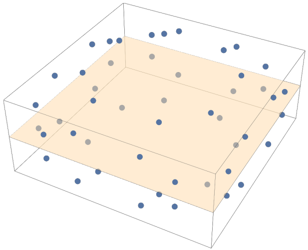
Picture a **finite transitive** subset of the unit sphere in $\mathbb{R}^d$

- Transitive under the orthogonal group ("every point looks the same")

What does it look like?

Can your set be fairly evenly spread out on the whole sphere?
(If allow infinite transitive sets, can take the whole sphere, or a countable dense subset)

Counterintuitively, it cannot! Finite transitive sets must lie close to some hyperplane



**Question.** (Z. '16)  (Motivated by expansion in Cayley graphs)
Do all **finite transitive** subset of the unit sphere in $\mathbb{R}^d$ have small width?

- The following set has width $O(1/\sqrt{\log d})$:

$$\left\{ \text{coordinate-permutations of } \frac{\sim 1}{\sqrt{\log d}} \left( \pm 1, \frac{\pm 1}{\sqrt{2}}, \ldots, \frac{\pm 1}{\sqrt{d}} \right) \right\} \subseteq \mathbb{R}^d$$

- I conjectured ('16) that this is essentially optimal

**Theorem.** (Green '20)
Every finite transitive subset of the unit sphere in $\mathbb{R}^d$ has width $O(\frac{1}{\sqrt{\log d}})$.

- The fact that it's $o_{d \to \infty}(1)$ was already new and surprising.
- Proof uses CFSG. Related to Jordan's theorem on finite linear groups.

**Question.** Does every finite transitive subset lie close to some higher codimensional subspace? (i.e., small "cylindrical width")

**Theorem.** (Sah–Sawhney–Z. '21) Every finite transitive subset of the unit sphere in $\mathbb{R}^d$ lies within distance $O(\frac{1}{\sqrt{\log(d/k)}})$ of some codim-$k$ subspace $\forall k \leq d/(\log d)^C$

**Conjecture.** True for all $k$.

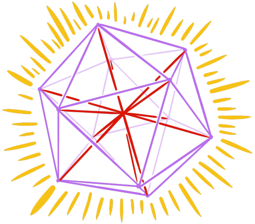
**Remark.** $O(\frac{1}{\sqrt{\log(d/k)}})$ best possible

**Conjecture.** ("cubical width") Every finite transitive set of unit vectors in $\mathbb{R}^d$ can be rotated so that all coordinates are $O(\frac{1}{\sqrt{\log d}})$

- Even $o(1)$ is open. Our thm $\implies$ can get first $d^{0.99}$ coordinates all $O(\frac{1}{\sqrt{\log d}})$

**Open problem.** (Width of small transitive sets)
Does every transitive set of $d^{O(1)}$ points on a unit sphere in $\mathbb{R}^d$ have width $O(1/\sqrt{d})$?

(Easy to get $\sqrt{(\log d)/d}$ using a random direction)



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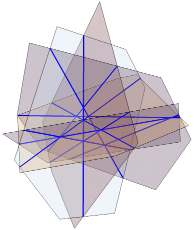

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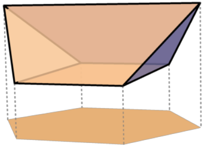

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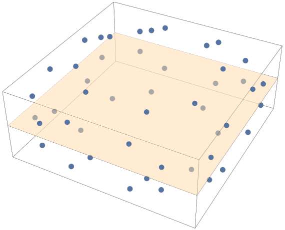

Width of transitive sets