Equiangular lines, spherical two-distance sets & spectral graph theory

Equiangular lines

\[ N(d) = \max \text{ # lines in } \mathbb{R}^d \text{ pairwise same angle} \]

E.g.  \[ N(2) = 3 \]
\[ N(3) = 6 \]

\[ \text{deCaen} \quad c \cdot d^2 \leq N(d) \leq \binom{d+1}{2} \]

\[ \text{angles} \to 90^\circ \quad \text{as } d \to \infty \quad \text{Gerzon '73} \]

Based on arXiv:1907.12466 & 2006.06633
Equiangular lines with a fixed angle

\[ N_\alpha(d) = \text{max } \# \text{equiangular lines with angle } \cos^{-1}(\alpha) \]

Some angles are more special

Lemmens-Seidel '73
\[ N_{\sqrt{3}}(d) = 2(d-1) \quad \forall d \geq 15 \]

Neumann '73
\[ N_\alpha(d) \leq 2d \quad \text{unless } \alpha = \frac{1}{\text{odd integer}} \]

Neumaier '89
\[ N_{\sqrt{5}}(d) = \left\lfloor \frac{3}{2} (d-1) \right\rfloor \quad \forall d \geq d_0 \]

Next interesting case \( \alpha = 1/7? \)

Finally, we remark that the recent result of Shearer [13], that every number \( t \geq t^* = (2 + \sqrt{5})^2 \approx 2.058 \) is a limit point from above of the set of largest eigenvalues of graphs, makes it likely that the hypothesis of Theorem 2.6 can be satisfied if and only if \( t < t^* \). (As communicated to me by Professor J. J. Seidel, Eindhoven, this has indeed been verified by A. J. Hoffman and J. Shearer.) Thus the next interesting case, \( t = 3 \), will require substantially stronger techniques.

Some decades later ...

Bukh '16
\[ N_\alpha(d) \leq C_\alpha d \]

Balla-Dräxler - Keevash-Sudakov '18
\[ N_\alpha(d) \leq 1.93 d \quad \forall d \geq d_0(\alpha) \quad \text{if } \alpha \neq 1/3 \]
Problem: determine, for each $\alpha$

$$\lim_{d \to \infty} \frac{N_{\alpha}(d)}{d}$$

Our work completely solves this problem

Lemmens-Seidel '73  \( N_{1/3}(d) = 2(d-1) \)  \( \forall d \text{ suff. large} \)

Neumaier '89  \( N_{1/5}(d) = \left\lfloor \frac{3}{2}(d-1) \right\rfloor \)

Our result:  \( N_{1/7}(d) = \left\lfloor \frac{4}{3}(d-1) \right\rfloor \)

**Theorem (JTYZZ)**  \( \forall \text{ integer } k \geq 2 \)

\[ N_{\frac{1}{2k-1}}(d) = \left\lfloor \frac{k}{k-1} (d-1) \right\rfloor \quad \forall d \geq d_0(k) \]

And for other angles  \( \forall \text{ fixed } \alpha \in (0,1) \)

Set  \( \lambda = \frac{1-\alpha}{2\alpha} \)

(reparameterization)  "spectral radius order"

Then

\[ N_{\alpha}(d) = \begin{cases} 
\left\lfloor \frac{k}{k-1} (d-1) \right\rfloor & \forall d \geq d_0(\alpha) \text{ if } k < \infty \\
 d + o(d) & \text{if } k = \infty
\end{cases} \]
\( k(\lambda) = \text{spectral radius order} \)

\[ = \min k \text{ s.t. } \exists k\text{-vertex graph } G \text{ with } \lambda_1(G) = \lambda \]

(set \( k(\lambda) = \infty \) if \( \nexists \) such \( G \) )  

spectral radius of \( G \)

= top eigenvalue

of adjacency mat of \( G \)

Examples

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<th>( \lambda )</th>
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<th>( k )</th>
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\[ \lim_{d \to \infty} \frac{N_\lambda(d)}{d} = \frac{k(\lambda)}{k(\lambda) - 1} \]

was conjectured by Jiang - Polyanskii

who proved it for \( \lambda < \sqrt{2+\sqrt{5}} \approx 2.058 \)

\( \{ \lambda_1(G) : G \in \mathbb{R} \} \)

Hoffman’72 + Shearer’89
Spherical two distance sets

A set of unit vectors in $\mathbb{R}^d$ whose inner products take only two values $\alpha, \beta$

(equiangular lines : $\alpha = -\beta$)

Delsarte, Goethals, Seidel '77 max size $\leq \frac{1}{2}d(d+3)$

Taking midpoints of a regular simplex $\sim \frac{1}{2}d(d+1)$

Glaizerin-Yu '18 : tight if $d > 7$ &

d + 3 not odd perf sq

From now on let's consider fixed angles

More generally: spherical A-code, $Ac[-1, 1)$

$N_A(d) = \max$ # unit vectors in $\mathbb{R}^d$

whose pairwise inner products lie in $A$

Equiangular lines $\sim A = \{ -\alpha, \alpha \}$

We consider $A = \{ \alpha, \beta \}$ for fixed $-1 \leq \beta < 0 < \alpha < 1$
Neumaier '81 \( N_{\alpha, \beta}(d) \leq 2d+1 \) unless \( \frac{1-\alpha}{\alpha-\beta} \in \mathbb{Z} \)

Bukh '16 \( N_{[-1, \beta]}^E \leq O(d) \)

Balla-Dräxler - Keevash-Sudakov '18
\[
N_{[-1, \beta]}^{\alpha_1, \alpha_2, \ldots, \alpha_k}(d) \leq 2^k(k-1)! \left(1 + \frac{\alpha_1}{\beta} + \cdots + \alpha_k\right) n^k
\]

& \exists \alpha_1, \ldots, \alpha_k, \beta \; s.t. \; \text{bound tight up to constant factor}

**Problem**
Determine, for fixed \(-1 \leq \beta < 0 < \alpha < 1\),
\[
\lim_{d \to \infty} \frac{N_{\alpha, \beta}(d)}{d}
\]

We conjecturally relate this problem to eigenvalues of signed graphs.

\[ G_{\alpha}^\pm \]
\[ A_{\alpha} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \]

**Defn** A valid \( t \)-coloring:
- + edges join identical colors
- - edges join distinct colors

\[
k_p(\lambda) = \inf \left\{ \frac{|G_{\alpha}^\pm|}{\text{mult}(\lambda, G_{\alpha}^\pm)} : G_{\alpha}^\pm \text{ has valid } p \text{-coloring} \right\}
\]

\[
\lambda_i(G_{\alpha}^\pm) = \lambda
\]
If $G^2$ has valid 2-coloring then $G^2$ is isospectral with its underlying graph.

Thus $k_1(\lambda) = k_2(\lambda) = k(\lambda) = \{ |G| : \lambda_i(\alpha) = \lambda \}$

Determining $k_p(\lambda)$, $p > 3$ seems hard.

Main conjecture on spherical two-distance sets:

Fix $-1 \leq \beta < 0 < \alpha < 1$. Set

$$\lambda = \frac{1 - \alpha}{\alpha - \beta} \quad \& \quad p = \left\lfloor \frac{1 - \alpha}{\beta} \right\rfloor + 1$$

Then

$$\lim_{d \to \infty} \frac{N_{\alpha, \beta}(d)}{d} = \frac{k_p(\lambda)}{k_p(\lambda - 1)} \quad (d = 1 \text{ if } k_p(\lambda) = \infty)$$

**Thm (JTYZZ)** Conjecture true if $p \leq 2$ or $\lambda \in \{1, \sqrt{2}, \sqrt{3} \}$.

$$k_p(1) = \begin{cases} 2 & \text{if } p = 1, 2 \\ \frac{p}{p - 1} & \text{if } p > 2 \end{cases}$$

$$k_p(\sqrt{2}) = \begin{cases} 3 & \text{if } p = 1, 2 \\ 2 & \text{if } p > 2 \end{cases}$$

$$k_p(\sqrt{3}) = \begin{cases} 4 & \text{if } p = 1, 2 \\ \frac{7}{3} & \text{if } p = 3 \\ 2 & \text{if } p > 4 \end{cases}$$

E.g. $(\alpha, \beta) = \left( \frac{2}{5}, \frac{-1}{5} \right)$, $\lambda = 1$, $p = 3$, $\nabla k_p(\lambda) = \frac{3}{2}$, $N_{\alpha, \beta}(d) = 3d + O(1)$ (contrasting $N_{\alpha, \beta}(d) \leq (2 + o(1))d$ for eq-ang lines)

cf. Huang pf. pf. sensitivity conjecture.
The problem that we are about to discuss is one of the founding problems of algebraic graph theory, despite the fact that at first sight it has little connection to graphs. A simplex in a metric space with distance function \(d\) is a subset \(S\) such that the distance \(d(x,y)\) between any two distinct points of \(S\) is the same. In \(\mathbb{R}^d\), for example, a simplex contains at most \(d + 1\) elements. However, if we consider the problem in real projective space then finding the maximum number of points in a simplex is not so easy. The points of this space are the lines through the origin of \(\mathbb{R}^d\), and the distance between two lines is determined by the angle between them. Therefore, a simplex is a set of lines in \(\mathbb{R}^d\) such that the angle between any two distinct lines is the same. We call this a set of equiangular lines. In this chapter we show how the problem of determining the maximum number of equiangular lines in \(\mathbb{R}^d\) can be expressed in graph-theoretic terms.

**Unit vectors** \(v_1, v_2, \ldots, v_N \in \mathbb{R}^d\) \(\langle v_i, v_j \rangle = \pm \alpha \ \forall i \neq j\)

**Associated graph** \(vtx [N]\), \(i \sim j\) if \(\langle v_i, v_j \rangle = -\alpha\) (obtuse)

**Observation:**

\(\exists N\) equiangular lines in \(\mathbb{R}^d\) with common angle \(\cos^{-1}\alpha\)

\(\exists N\)-vtx graph \(G\) st. \(\lambda I - A_G + \frac{1}{2}J\) is positive semidef

\[\lambda = \frac{1 - \alpha}{2 + \alpha}\]

\[J = \text{All 1}\]

and rank \(\leq d\)

\[= \frac{1}{2d}\text{ Gram matrix}\]

Recall **GOAL**: maximize \(N\) given \(\lambda\), \(d\)

**Construction**

Starting with some \(H\) with \(\lambda_1(H) = \lambda\) \& \(|H| = k(\lambda)\)

take \(G = H \bigoplus H \bigoplus H \ldots \bigoplus H \bigoplus H\) (\(N\) vtx total)
Upper bound on $N$

$N = \text{rank} \left( \lambda I - A_G + \frac{1}{2} J \right) + \text{null} \left( \lambda I - A_G + \frac{1}{2} J \right)$

$\leq \ d \ + \ \text{null} \left( \lambda I - A_G + \frac{1}{2} J \right)$

$\leq \ d \ + \ \text{null} \left( \lambda I - A_G \right) \ + \ \text{mult} (\lambda, G) \ \text{eigenvalue multiplicity}$

Difficult/interesting case to rule out:

- a large connected $G$ with high $\text{mult} (\lambda, G)$

Note that since $\lambda I - A_G + \frac{1}{2} J$ is psd

$\lambda = 1^{st} \ or \ 2^{nd}$ largest eigenvalue of $A_G$

If $\lambda = \lambda_1 (G)$, Perron–Frobenius $\Rightarrow$ $\text{mult} (\lambda, G) = 1$

So focus on the case $\lambda = \lambda_2$

Q: Must all connected graphs have small 2nd eigenvalue multiplicity?
Not all graphs can arise from equiangular lines

Switching operation

\[
\begin{align*}
&\text{Thm (Balla-Dräxler-Keevash-Sudakov)} \\
&\forall \Delta \exists G : \text{can switch } G \text{ to } \max \deg \leq \Delta
\end{align*}
\]

We prove a new result in spectral graph theory:

\[
\boxed{\text{Thm [JTYZZ]} \ A \text{ connected } n\text{-vertex graph with } \max \deg \leq \Delta \text{ has second largest eigenvalue with multiplicity } O_\Delta\left(\frac{n}{\log \log n}\right)}
\]

Near miss examples

- Strongly regular graphs
  e.g. complete graphs, Paley graphs
  Not bounded degree
- \[\begin{array}{c}
\vdots \quad \ddots \quad \vdots \quad \ddots \quad \vdots \\
\end{array}\]
  mult \((0,G)\) linear, \(0\) a middle eigenvalue
- \[\begin{array}{c}
\triangle \quad \triangle \quad \square \quad \square \quad \square \\
\end{array}\]
  not connected
Open problem: Max. possible 2nd eigenvalue multiplicity of a connected bounded degree graph?

Interesting to consider restrictions to (bdd deg)
  - regular graphs
  - Cayley graphs

Example: a Cayley graph on $\text{PSL}(2, p)$ gives 2nd eigenvalue multiplicity $\geq n^{1/3}$

For expander graphs, $\text{mult}(\lambda_2, G) = O\left(\frac{n}{\log n}\right)$

For non-expanding Cayley graphs, $\text{mult}(\lambda_2, G) = O(1)$

Lee-Makarychev, building on Gromov, Colding-Minicozzi, Kleiner

Recently: McKenzie-Rasmussen-Srivastava
  for a connected $d$-reg graph, $\text{mult}(\lambda_2, G) \leq O_d\left(\frac{n}{\log^{d-1} n}\right)$

Let's prove:

Thm: If $G$ is connected, $n$ vtx, max deg $\leq \Delta$ then its 2nd largest eigenvalue has multiplicity $O\left(\frac{n}{\log \log n}\right)$
**Lem 1** (Finding a small net)

Every connected \( n \)-vtx graph has an \( r \)-net of size \( \frac{n}{r+1} \), \( \forall n, r \).

**Proof** (PF)

1. Select a spanning tree
2. Pick an arbitrary root
3. Walk back \( r \) steps
4. Add to net & truncate, repeat

**Lem 2** (Net removal significantly reduces spectral radius)

If \( H = G - (\text{an } r\text{-net of } G) \), then \( \lambda_1(H)^{2r} \leq \lambda_1(G)^{2r} - 1 \).

**Proof** (PF)

\( A_H^{2r} \leq A_G^{2r} - I \) entrywise (\( A_H = A_G \) with the deleted edges zero'd)

To check diagonal entries, count closed walks
Suffice to exhibit a closed walk \( v \Theta \) in \( G \) not in \( H \).

**Lem 3** (Local versus global spectra)

\[
\frac{|H|}{\sum_{i=1}^{\lambda_i(H)^{2r}} \leq \sum_{v \in V(H)} \lambda_1(B_H(v,r))^{2r}}
\]

**Proof** (PF)

\# closed walks of length \( 2r \) in \( H \)

\# such walks starting at \( v \) (necessarily stays in \( B_H(v,r) \))

\[= 1^T A_B^{2r} 1_v \]

\[\leq \lambda_1(B_H(v,r))^{2r}\]
Tool: Cauchy eigenvalue interlacing theorem

Real sym matrix $A$, then eigenvalues of $A$ & $A'$ interlace

Remove last row & column $\rightarrow A'$

$\Rightarrow$ Deleting a vertex cannot reduce $\text{mult}(\lambda, G)$ by more than 1

Proof sketch that $\text{mult}(\lambda_2, G) = o(n)$

Let $r = r_1 + r_2$, $r_1 = c \log \log n$, $r_2 = c \log n$

$U = \{ v \in V(G) : \lambda_1(B_G(v, r)) > \lambda \}$

(vtx with large local spectral radius)

If $U$ contains $u, v$ with $d(u, v) \geq 2r + 2$ then $G$ restricted to these two balls has $\geq 2$ eigenval $> \lambda$ Contradiction.

Thus $U \subset a (2r+1)$-ball $\Rightarrow |U| \leq \Delta^{2r+2} = o(n)$

Net removal: let $V_0$ be an $r_1$-net of $G$ with $|V_0| \leq \lceil \frac{n}{r_1} \rceil$ by Lem 1
Set $H = G - U \cup V_0$.

For all $v \in V(H)$, $B_G(v, r) \setminus B_H(v, r_2)$ is an $r_i$-net of $B_G(v, r)$.

By Lemma 2, for all $v \in V(H)$,

$$\lambda_1(B_H(v, r_2))^{2r_1} \leq \lambda_1(B_G(v, r))^{2r_1} - 1 \leq \lambda^{2r_i} - 1 \quad \text{(since } v \notin U)$$

By Lemma 3, \[ \frac{|H|}{\sum_{i=1}^{n} \lambda_i(H)}^{2r_2} \leq \sum_{v \in V(H)} \lambda_1(B_H(v, r_2))^{2r_2} \]

$$\text{mult}(\lambda, H)^{2r_2} \leq \frac{1}{(\lambda^{2r_i} - 1)^n}$$

$$\Rightarrow \text{mult}(\lambda, H) = o(n)$$

By interlacing, $\text{mult}(\lambda, G) \leq \text{mult}(\lambda, H) + |U| + |V| = o(n)$

**Summary:**
- Bound moment by counting closed $2r_2$-walks
- Net removal significantly reduces local closed $2r_i$-walks
- Relate these via local spectral radii
Spherical 2-dist set: unit vectors $v_1, \ldots, v_N \in \mathbb{R}^d$ (fixed $-1 < \beta < 0 < \alpha < 1$) $\langle v_i, v_j \rangle \in \{\alpha, \beta\}$ $\forall i \neq j$

Associated graph $G$: $i \sim j$ if $\langle v_i, v_j \rangle = \beta$ (obtuse)

“Switching” no longer valid $\quad \sim \quad \not\sim$

**Structure theorem**

After deleting $O(1)$ vtx, $G$ can be modified into a complete $p$-partite graph $\mathcal{P}$ where $O(1)$ edges are added/removed at each vtx

We would be able to proceed the same as eq-arg lines if all such signed graphs have $\text{mult}(\mathcal{P}, G) = o(n)$

But this is not true
\exists G_n^\pm \text{ signed graph on } n \text{ vertices max deg } 5 \text{ has a valid } 3\text{-coloring. But: largest eigenvalue appears with multiplicity } n

But there is still hope.

Our structure theorem actually gives additional forbidden subgraphs not mentioned above.

To solve the spherical 2-dist set problem for \( \lambda = \frac{1-\delta}{\alpha - \beta} \text{ and } p = \left\lfloor \frac{-\delta}{\beta} \right\rfloor + 1 \), it suffices to

1. Determine

\[
k_p(\lambda) = \inf \left\{ \frac{|G|}{\text{mult}(\lambda, G)} : G \text{ has valid } p\text{-coloring and } \lambda(G^\pm) = \lambda \right\}
\]

2. Give a sufficiently good linear upper bound on \( \text{mult}(\lambda_{\text{pt}}, G) \) for certain classes of signed graphs \( G^\pm \) given by forbidden subgraphs.
Next interesting case $\lambda = 2$ (we solved $\lambda \in \{1, \sqrt{5}, \sqrt{15}\}$)

Paley graph of order $9$

$\Rightarrow k_3(2) \leq \frac{9}{4}$

Clebsch graph

$\Rightarrow k_4(2) \leq \frac{8}{5}$