Last time

Distinct distances theorem (Guth-Katz)

$P \subset \mathbb{R}^2$, $|P| = N$. Then P determines $\gtrsim \frac{N}{\log N}$ distinct distances

Partial symmetries (Elekes-Sharir framework)

$G_r(P) = \{ g \in G : |g(P) \cap P| \geq r$

G: rigid motions in the plane

Key theorem $P \subset \mathbb{R}^2$, $|P| = N$, $\forall r \geq 2$

$$|G_r(P)| \lesssim \frac{N^3}{r^2}$$

After straightening the coordinates,

$\forall p, q \in \mathbb{R}^2$

remove translations

$$\{ g \in G' : g(p) = q \}$$

is represented as a straight line $l_{p,q} \subset \mathbb{R}^3$

$$\mathcal{L}(P) = \{ l_{p,q} \}_{\substack{p,q \in P \\ p \neq q}}$$

Would like to show that for $\mathcal{L} = \mathcal{L}(P)$

$$|P_r(\mathcal{L})| \lesssim \frac{|\mathcal{L}|^{3/2}}{r^2}$$

$P_r(\mathcal{L})$: r-rich pts

i.e., pts in $\geq r$ lines in $\mathcal{L}$

ST in $\mathbb{R}^2$: $|P_r(\mathcal{L})| \lesssim \frac{L^2}{r^3} + \frac{L}{r}$

<u>Lem</u> $\mathcal{L}(P)$ contains $O(N)$ lines in any plane or deg 2 surface.

<u>Pf</u> (for plane).

For a fixed p, any two lines $l_{p,q}$, $l_{p,q'}$ are skew.

- disjoint : if $g(p) = q$, then $g(p) \neq q'$
- not parallel : same argument for translations

For each $p \in P$, only one of $\{l_{p,q}\}_q$ can lie in a given plane. So $\leq N$ lines from $\mathcal{L}(P)$ in this plane

Key incidence estimate

<u>Thm</u> (Guth-Katz). $\mathcal{L}$: L lines in $\mathbb{R}^3$

(a) If $\leq L^{1/2}$ lines in any plane or deg 2 surface then $|P_2(\mathcal{L})| \lesssim L^{3/2}$

(b) If $\leq L^{1/2}$ lines in any plane, then $|P_r(\mathcal{L})| \lesssim \frac{L^{3/2}}{r^2}$ $3 \leq r \leq 2L^{1/2}$

Reguli

Prop For any 3 lines in $\mathbb{F}^3$, there is a non-zero polynomial of deg ≤ 2 vanishing on them.

Pf Pick 3 pts on every line,

$\dim \mathrm{Poly}_2(\mathbb{F}^3) = 10$

$\exists P_{\not\equiv 0} \in \mathrm{Poly}_2(\mathbb{F}^3)$ vanishing on these 9 pts.

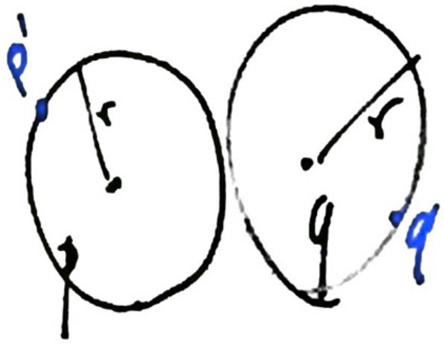
Since $\deg P \leq 2$, & vanish at 3 pts on l, it vanishes on l.

Prop. If l_1, l_2, l_3 are pairwise skew lines in $\mathbb{F}^3$, then there is an irreducible alg surface $R(l_1, l_2, l_3)$ which contains every line that intersects l_1, l_2, l_3.

$\boxed{\text{Regulus}}$

Example with reguli

$S^1(q,r)$: circle around q of radius r



$\{g \in G' : g(p) \in S^1(q,r)\} = R$

First ruling: $\{\ell_{p,q'} : q' \in S^1(q,r)\}$

Second ruling $\{\ell_{p',q} : p' \in S^1(p,r)\}$

Every $g \in R$ lies on one line from each ruling.

Thm $\exists$ const K.

If $\mathcal{L}$ is a set of L lines in $\mathbb{R}^3$, with $|P_3(\mathcal{L})| \geq K L^{3/2}$, then there is a plane that contains $\geq 10 L^{1/2}$ lines of $\mathcal{L}$.

Idea: Combinatorial structure.

$\downarrow$ polynomial P of low deg vanishing on $\mathcal{L}$

Alg structure.

$\downarrow$ $Z(P)$ has many flat points

Geometric structure

[Structural result]

Cor If $\mathcal{L}$ a set of L lines in $\mathbb{R}^3$, then there is a set of planes $\Pi_1, \ldots, \Pi_S$, $S \leq L^{1/2}$, and disjoint $\mathcal{L}_i \subset \mathcal{L}$ so that $\mathcal{L}_i$ contains in Π_i, and

$$\left| P_3(\mathcal{L}) \setminus \bigcup_i P_3(\mathcal{L}_i) \right| \leq K L^{3/2}$$

Degree reduction

Recall: (a) $S \subset \mathbb{F}^n$, $|S| < \dim \mathrm{Poly}_D(\mathbb{F}^n) = \binom{D+n}{n}$

then $\underbrace{\deg(S)}$ $\leq D$

lowest degree of non-zero poly vanishing on S

$\deg(S) \leq n|S|^{1/n}$

(b) $\mathcal{L}$: L lines in $\mathbb{F}^n$

If $(D+1)L < \dim \mathrm{Poly}_D(\mathbb{F}^n) = \binom{D+n}{n}$

then $\deg(\mathcal{L}) \leq D$.

$\deg(\mathcal{L}) \leq (2n+1) L^{\frac{1}{n-1}}$

Prop $\mathcal{L}$ a set of lines $\mathbb{F}^3$

Each line of $\mathcal{L}$ contains $\geq A$ pts of $P_2(\mathcal{L})$

Then $\deg(\mathcal{L}) \lesssim \frac{L}{A}$

Interesting when $A \gg \sqrt{L}$

Bound cannot be improved below $\frac{L}{A+1}$ when $A \geq \sqrt{L}$

Take $\frac{L}{A+1}$ planes.

$A+1$ lines in gen pos in each plane.

Take prod of linear poly Q_i gets us deg $\frac{L}{A+1}$

Cannot do better.

If P vanishes on $\mathcal{L}$. by B.T.

either $\deg P \geq A+1$

or $Q_i \mid P \Rightarrow \deg P \geq \min\left\{A+1, \frac{L}{A+1}\right\}$.

<u>Bezout's Thm</u> $P, Q \in \mathrm{Poly}(\mathbb{F}^2)$ no common factor, then

$$Z(P,Q) \leq (\deg P)(\deg Q).$$

<u>Bezout's Thm for lines</u>

$\mathbb{F}$ infinite field. $P, Q \in \mathrm{Poly}(\mathbb{F}^3)$. no common factor. Then $Z(P,Q)$ has $\leq (\deg P)(\deg Q)$ lines.

When $A \geq L^{1/2}$.

$\deg P \geq \frac{L}{A+1}$

Contagious vanishing argument

Idea: by param. find poly deg $D \approx \frac{L}{A}$ that vanishes on D^2 lines of $\mathcal{L}$

Interesting case $D^2 \ll L$

- Vanishing is "contagious"

  $\rightarrow$ spreads to other lines

Initially use P to kill. D^2 random lines "infected"

For each $\ell \in \mathcal{L}$, expected $\gtrsim A \cdot \frac{D^2}{L}$ infected intersection pts.

If $> 10D$, expect P to vanish on most lines in $\mathcal{L}$.

Tail bound $X \sim Bin(N, p)$

$$\mathbb{P}(|X| > 2pN) \leq \exp\left(-\frac{pN}{100}\right).$$

$$\mathbb{P}(|X| < \tfrac{1}{2}pN) \leq \exp\left(-\frac{pN}{100}\right).$$

Pf of prop. $p = \frac{1}{20}\frac{D^2}{L}$

Inf. each line w.p p.

whp # inf lines $\leq \frac{1}{10} D^2$

Find a poly deg $\lesssim D$

Vanishing on the infected lines.

Fix $\ell \in \mathcal{L}$. Expected # inf. pts

on ℓ is $\geq Ap = \frac{1}{20}\frac{D^2 A}{L} \geq 10^4 D$

$D \approx 10^6 \frac{L}{A}$

If ℓ has $> D$ inf. pts,

then $P = 0$ on ℓ.

This occurs w.p. $\geq 1 - e^{-100 D}$

$\geq 1 - e^{-10^7 L/A}$

If $\frac{L}{A} > 10^{-5} \log L$, then whp P

vanishes on every line.

If $\frac{L}{A} \leq 10^{-5} \log L$, whp P vanishes

$\geq \frac{99}{100} L$ lines of $\mathcal{L}$

Induction on L.

Finish by induction on L.

Induction hypothesis: $\deg(\mathcal{L}) \leq 10^7 \frac{L}{A}$

Sketch: $\exists P_1$ vanishes on $\mathcal{L}_1 \subset \mathcal{L}$, $|\mathcal{L}_1| \geq \frac{99}{100} L$

does not vanish on $\mathcal{L}_2$, $|\mathcal{L}_2| \leq \frac{1}{100} L$

Each lines intersects $Z(P)$ at $\leq \deg(P_1)$

So it has at $\geq A - \deg(P_1)$

intersection pts with other lines in $\mathcal{L}_2$.

$A \geq \frac{10^5 L}{\log L}$, $\deg P_1 \leq 10^6 \frac{L}{A} \leq 10 \log_2 L$

So

$\geq \frac{9}{10} A$.

By induction,

$$\deg(\mathcal{L}_2) \leq 10^6 \frac{L}{A}$$

So

$$\deg(\mathcal{L}) \leq \deg(\mathcal{L}_1) + \deg(\mathcal{L}_2)$$

$$\leq \frac{10^6 L}{A} + \frac{10^6 L}{A}$$

$$\leq \frac{10^7 L}{A}.$$