Last time:

Polynomial Partitioning theorem

Fix $n \geq 2$. Let $X \subset \mathbb{R}^n$, $D > 0$.

Then there is a nonzero polynomial $P \in \mathrm{Poly}_D(\mathbb{R}^n)$ s.t. $\mathbb{R}^n \setminus Z(P)$ is a disjoint union of $\lesssim D^n$ open sets, each containing $\lesssim \frac{|X|}{D^n}$ points of X.

We proved it via repeated applications of the polynomial ham sandwich theorem.

Szemerédi - Trotter theorem

$\mathfrak{S}$: S pts in $\mathbb{R}^2$, $\mathcal{L}$: L lines in $\mathbb{R}^2$

$$I(\mathfrak{S}, \mathcal{L}) \lesssim S^{2/3} L^{2/3} + S + L.$$

Simple estimate:

$$I(\mathfrak{S}, \mathcal{L}) \leq L + S^2$$

$$I(\mathfrak{S}, \mathcal{L}) \leq S + L^2$$

lines with ≤ 1 pt $\quad \leq L$

lines with ≥ 2 pts $\quad \leq S^2$

Pf of ST

Assume: $S^{1/2} \le L \le S^2$ (otherwise ST follows from simple estimate)

Choose D later.

Apply poly part.: poly P, $\deg \le D$

each cell contains $\lesssim \frac{S}{D^2}$ points of $\mathfrak{S}$

Some of the pts could lie on $Z(P)$.

$\mathfrak{S} = \mathfrak{S}_{alg} \cup \mathfrak{S}_{cell}$ — $\mathfrak{S}_{alg}$: pts of $\mathfrak{S}$ on $Z(P)$; $\mathfrak{S}_{cell}$: other pts

$\mathfrak{L} = \mathfrak{L}_{alg} \cup \mathfrak{L}_{cell}$ — $\mathfrak{L}_{alg}$: lines of $\mathfrak{L}$ lying in $Z(P)$; $\mathfrak{L}_{cell}$: other lines.

$\mathfrak{S}_{cell} = \bigcup \mathfrak{S}_i$ ← pts in the i-th cell

$\mathfrak{L}_{cell} = \bigcup \mathfrak{L}_i$ ← lines that pass thru the i-th cell

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Any line in $\mathfrak{L}_{cell}$ meets $Z(P)$ at $\le D$ pts. thus enters $\le D+1$ cells.

$$\sum L_i \le (D+1) L \qquad (L_i = |\mathfrak{L}_i|)$$

Apply easy bound:

$$I(\mathfrak{S}_i, \mathfrak{L}_i) \le L_i + S_i^2$$

$$I(\mathfrak{S}_{cell}, \mathfrak{L}) = \sum_i I(\mathfrak{S}_i, \mathfrak{L}_i)$$

$$\le \sum_i (L_i + S_i^2)$$

$$\lesssim LD + \frac{S}{D^2} \sum_i S_i$$

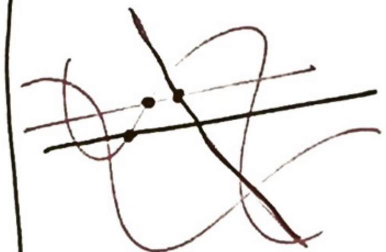
$$\le LD + \frac{S^2}{D^2}$$

Let's deal with $\mathcal{S}_{alg}$

$$|I(\mathcal{S},\mathcal{L})| \le |I(\mathcal{S}_{cell},\mathcal{L})| + |I(\mathcal{S}_{alg},\mathcal{L}_{cell})| + |I(\mathcal{S}_{alg},\mathcal{L}_{alg})|$$

- Each line in $\mathcal{L}_{cell}$ meets $Z(P)$ at $\le D$ pts. Thus $|I(\mathcal{S}_{alg},\mathcal{L}_{cell})| \le LD$.
- $|\mathcal{L}_{alg}| \le D$

  $|I(\mathcal{S}_{alg},\mathcal{L}_{alg})| \le S + D^2$



$$\le LD + \frac{S^2}{D^2} + LD + S + D^2$$

Choose $D \simeq S^{2/3}L^{-1/3}$

$$\le S^{2/3}L^{2/3} \quad //$$

Since $S^2 \ge L$, $D \ge 1$

$D^2 \simeq S^{4/3}L^{-2/3} \le S$ by $L \ge S^{1/2}$

Distinct distance theorem (Guth-Katz)

$P \subset \mathbb{R}^2$, $|P| = N$.

Then P determines $\gtrsim \dfrac{N}{\log N}$ distinct distances

Partial symmetries (Elekes-Sharir)

G = gp of orientation-preserving rigid motions of the plane.

$$G_r(P) = \{ g \in G : \text{s.t. } |g(P) \cap P| \geq r \}$$

For a generic set of N pts,

$$|G_r(P)| = \begin{cases} \binom{N}{2} + 1 & r = 2 \\ 1 & r \geq 3 \end{cases}$$

Square grid $\sqrt{N} \times \sqrt{N}$

$$|G_r(P)| \approx \frac{N^3}{r^2} \qquad \forall\, 2 \leq r \leq \frac{N}{2}$$

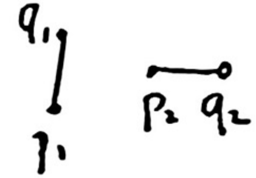
Thm (ES for $r=3$, GK for all $r \geq 2$)

$P \subset \mathbb{R}^2$, $|P| = N$

$$|G_r(P)| \lesssim \frac{N^3}{r^2} \qquad \forall\, r \geq 2$$

Let $d(P)$ be the set of distance

$$Q(P) := \{ (p_1, q_1, p_2, q_2) \in P^4 : |p_1 - q_1| = |p_2 - q_2| \neq 0 \}$$



Lem $P \subset \mathbb{R}^2$, $|P| = N$

$$|d(P)|\,|Q(P)| \geq (N^2 - N)^2$$

Pf Let the i-th dist d_i occur n_i times as $|p-q|$, $p,q \in P$

$$|Q(P)| = \sum_{j=1}^{|d(P)|} n_j^2$$

$$\overset{C\text{-}S}{\geq} \frac{1}{|d(P)|}\left(\sum n_j\right)^2$$

$$= \frac{1}{|d(P)|}(N^2 - N)^2$$

/

Prop $|Q(P)| = \sum_{r \geq 2} (2r-2)\,|G_r(P)|$

Using $|G_r(P)| \lesssim \frac{N^3}{r^2}$.

$$|Q(P)| \asymp \sum_{r=2}^{N} r\,|G_r(P)| \lesssim \sum_{r=2}^{N} \frac{N^3}{r} \sim N^3 \log N$$

Apply Lem: $|d(P)| \gtrsim \frac{N}{\log N}$

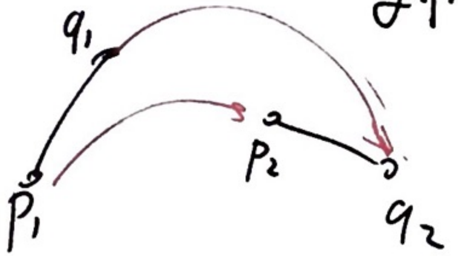
Prop $|Q(P)| = \sum_{r \geq 2} (2r-2) |G_r(P)|$

Pf $E : Q(P) \to G_2(P)$

sending $(p_1, q_1, p_2, q_2) \in Q$

to the unique $g \in G$ s.t.

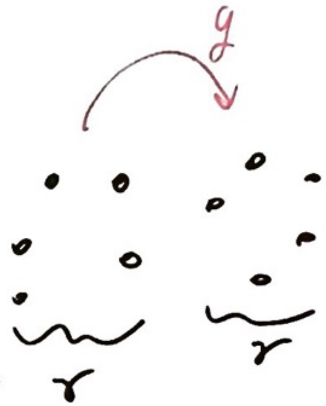
$g(p_1) = p_2, \quad g(q_1) = q_2$.



E is not injective

If $|g(P) \cap P| = r$,

then $|E^{-1}(g)| = 2\binom{r}{2}$



Write $G_{=r}(P)$

$= \{ g \in P : |g(P) \cap P| = r \}$.

$$Q(P) = \sum_{r=2}^{|P|} 2\binom{r}{2} |G_{=r}(P)|$$

$$= \sum_{r=2}^{|P|} 2\binom{r}{2} \left(|G_r(P)| - |G_{r+1}(P)| \right)$$

$$= \sum_{r \geq 2} |G_r(P)| \left(2\binom{r}{2} - 2\binom{r-1}{2} \right)$$

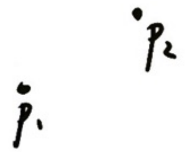
$$= \sum_{r \geq 2} (2r-2) |G_r(P)|$$

Incidence geometry

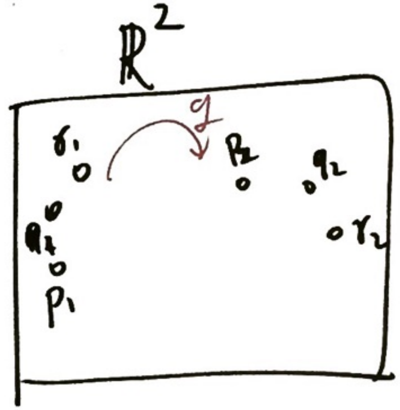
$p_1, p_2 \in \mathbb{R}^2$

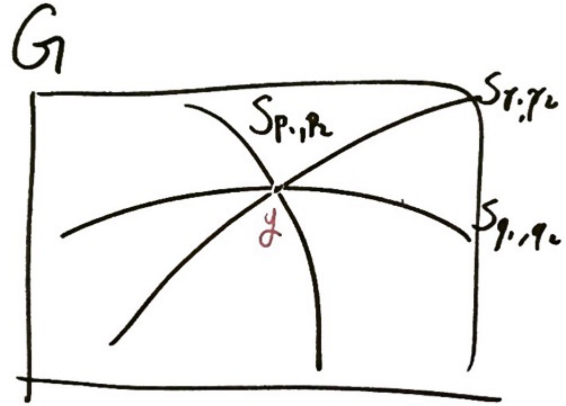
$$S_{p_1,p_2} = \{ g \in G : g(p_1) = p_2 \}$$

1-dim smooth curve in G.
diffeomorphic to a circle — 3-dim.



$G_r(P)$ is exactly the set of $g \in G$ that lie in $\geq r$ pts of the curves $\{S_{p_1,p_2}\}_{p_1,p_2 \in P}$





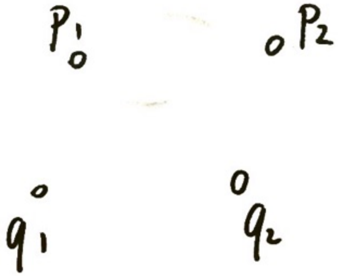
Straighten the coordinates of G so that S_{p_1,p_2} are lines

$G^{trans} \subset G$ be the translations.

Lem $P \subset \mathbb{R}^2$, $|P| = N$. Then

$$|G_r(P) \cap G^{trans}| \lesssim \frac{N^3}{r^2}$$

Pf The # of quadruples



(p_1, q_1, p_2, q_2)

$g(p_1) = p_2$,

$g(q_1) = q_2$, $g \in G^{trans}$

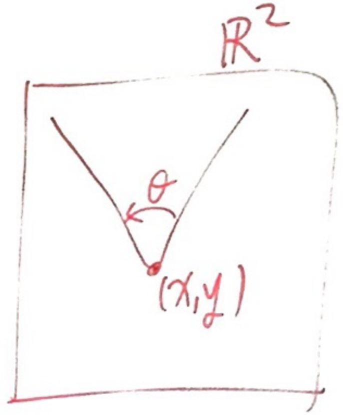
is $\leq N^3$.

And $2\binom{r}{2}$ of such quadruples are associated to each $G_r(P) \cap G^{trans.}$

So $|G_r(P) \cap G^{trans}| \leq \frac{N^3}{2\binom{r}{2}} \lesssim \frac{N^3}{r^2}$.

$G' = G \setminus G^{trans}$ rotations

Define: $\rho : G' \to \mathbb{R}^3$

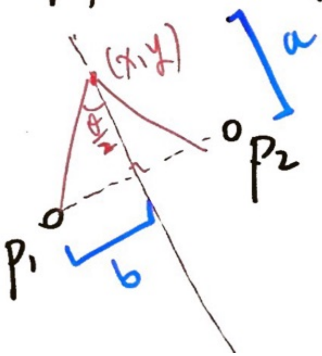


$\rho(g) = (x, y, \cot\frac{\theta}{2})$.

bijection.

Lem $\rho(S_{p_1,p_2} \cap G')$ is a line ℓ_{p_1,p_2} in $\mathbb{R}^3$.

Pf sketch



$\cot\frac{\theta}{2} = \frac{a}{b}$

Lem The lines $\{l_{p_1,p_2}\}_{p_1,p_2 \in \mathbb{R}^2}$ are all distinct (they represent different sets of rigid motions)

Lem $|G_r(P) \cap G'| = |P_r(\mathcal{L}(P))|$

let $\mathcal{L}(P) = \{l_{p_1,p_2}\}_{p_1,p_2 \in P}$

$P_r(\mathcal{L}) \leftarrow$ r-rich pts

pts in $\geq r$ lines in $\mathcal{L}$

Would like to prove:

$$|P_r(\mathcal{L}(P))| \lesssim \frac{N^3}{r^2} \approx \frac{|\mathcal{L}(P)|^{3/2}}{r^2}$$

Q Max # of r-rich pts in a set of L lines.

Want: $\lesssim \frac{L^{3/2}}{r^2}$

This can fail if the lines cluster on some plane or deg 2 surface

e.g. grid construction gives L lines with $\frac{L^2}{r^3}$ r-rich points

Lem $\mathcal{L}(P)$ contains $\lesssim N$ lines in any plane or deg 2 surface