Incidence Geometry.

$\mathcal{L}$ - set of L lines in $\mathbb{R}^2$

$\mathcal{S}$ - set of S points in $\mathbb{R}^2$

Q: max # of incidences between $\mathcal{L}$ and $\mathcal{S}$? (p,ℓ) $p\in\ell$

$$I(\mathcal{S},\mathcal{L}) = \{(p,\ell) : p \in \mathcal{S}, \ell \in \mathcal{L}, p \in \ell\}$$

$P_r(\mathcal{L})$ - set of points that lie in $\geq r$ lines "r-rich pts"

Q max # of r-rich pts? (in terms of L & r)

Szemerédi-Trotter theorem

$$|I(\mathcal{S},\mathcal{L})| \lesssim L^{2/3} S^{2/3} + L + S.$$

Equivalent formulation of ST:

$$|P_r(\mathcal{L})| \lesssim \frac{L^2}{r^3} + \frac{L}{r}$$

Exer These two versions are equiv.

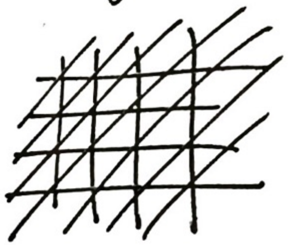
Examples.

If $r \geq \sqrt{L}$

$\frac{L}{r}$ pts, each in r lines.

$r = 2$ ✓

$r = 3$

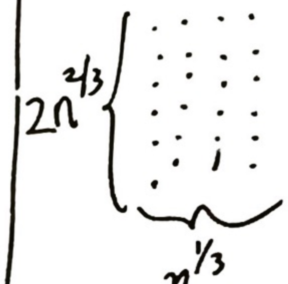


For larger r.

slopes where $\frac{a}{b}$. $a, b \lesssim \sqrt{r}$

take slopes the first r numbers in the list $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4} \cdots$

$S = L = n$



$\mathcal{L}$: $y = mx + b$

$1 \leq m \leq n^{1/3}, 1 \leq b \leq n^{2/3}$

$\#$ incidences $\approx n \cdot n^{1/3}$

$= n^{4/3}$

$\mathbb{F}_q^2$ $N = q^2$ pts

$N = q^2$ lines

$N^{3/2} = q^3$ incidence

Easy bound

Lem. $I(\mathcal{S},\mathcal{L}) \lesssim S L^{1/2} + L$

$I(\mathcal{S},\mathcal{L}) \lesssim L S^{1/2} + S$

Pf $|I(\mathcal{S},\mathcal{L})|^2 = \left(\sum_{\ell\in\mathcal{L}} |\ell\cap S|\right)^2$

$\overset{(C-S)}{\le} L \cdot \sum_{\ell\in\mathcal{L}} |\ell\cap \mathcal{S}|^2$

$= L\cdot \sum_{\ell\in\mathcal{L}} \left(|\ell\cap S| + |\{(p,q)\in \ell\cap\mathcal{S} : p\neq q\}|\right)$

$= L\cdot\left(I(\mathcal{S},\mathcal{L}) + S^2\right)$

$I(\mathcal{S},\mathcal{L}) \lesssim S L^{1/2} + L$

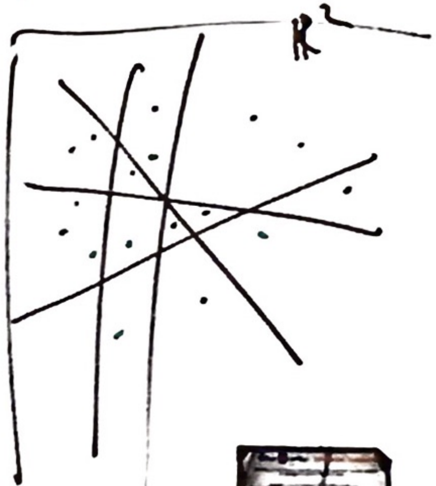
Aside C_4-free graphs

no 

max # edges in C_4-free graph

Cutting method.

- Cut the plane into pieces (by lines)
- Apply the easy bound to each cell.
- Aggregate



Heuristics

- D auxiliary lines to cut.
- cut into $\approx D^2$ components
- each $\ell \in \mathcal{L}$ enters $\leq D+1$ cells

Optimistic: suppose that the lines $\mathcal{L}$ and points $\mathcal{S}$ are distributed evenly across the cells

- Avg cell contains $\lesssim \frac{S}{D^2}$ pts of $\mathcal{S}$
  - Would be nice if all cells contains $\leq 1000 \frac{S}{D^2}$ pts of $\mathcal{S}$
- Avg cell .... $\lesssim \frac{L}{D}$ lines of $\mathcal{L}$
  - Would be nice ...... $\lesssim 1000 \frac{L}{D}$ lines of $\mathcal{L}$

Use the easy bound:

\# incidences in each cell

$$\lesssim \left(\frac{S}{D^2}\right)\left(\frac{L}{D}\right)^{1/2} + \frac{L}{D}$$

Add up across D^2 cells

$$\frac{SL^{1/2}}{D^{1/2}} + LD$$

Choose $D \sim S^{2/3} L^{-1/3}$. Get $S^{2/3} L^{2/3}$.

Only interesting case is $S^{1/2} \leq L \leq S^2$

How to evenly distribute by cutting

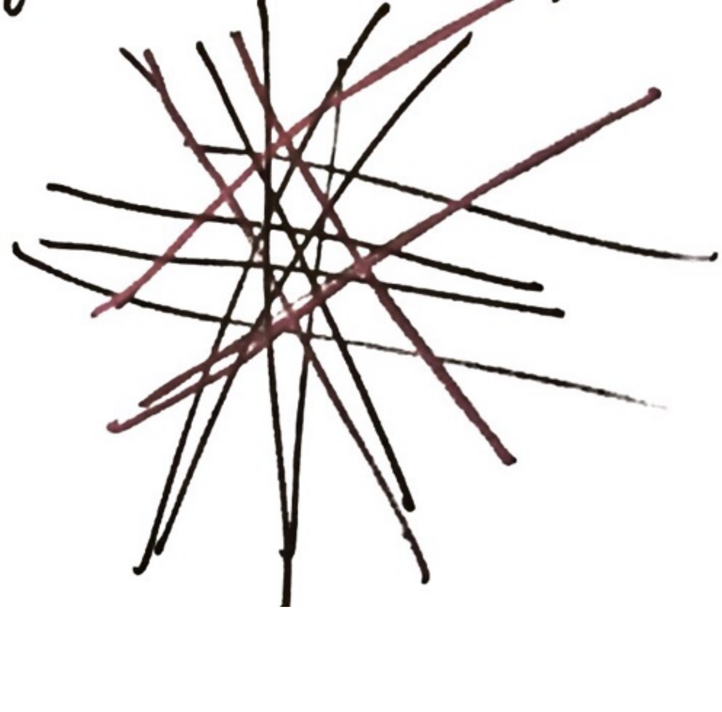
D lines

these pts fall into $\leq 2D$ cells.

2D cuts on the unit curve.

Strategy of the cellular method

- Choose the cutting lines randomly from $\mathcal{L}$



Polynomial Partitioning thm

$X \subset \mathbb{R}^n$, $D > 0$.

Then there is a polynomial $0 \neq P \in \mathrm{Poly}_D(\mathbb{R}^n)$

$\mathbb{R}^n \setminus Z(P)$ is a disjoint union of $\lesssim D^n$ open sets and each of them contains $\leq C(n)\frac{|X|}{D^n}$ points of X.

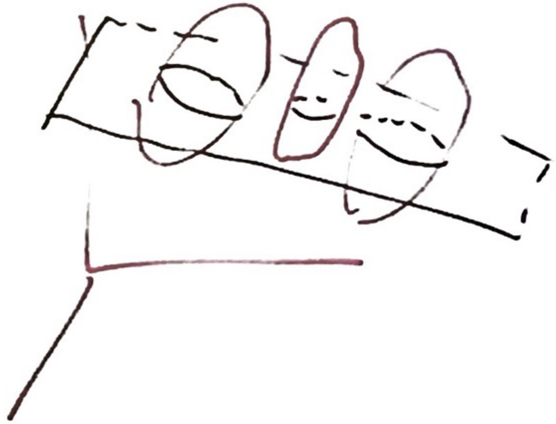
Caveat: some or all of the points of X may lie on $Z(P)$.

$\dim \mathrm{Poly}_D(\mathbb{R}^n) \simeq D^n$

Ham Sandwich Theorem (Stone-Tukey)

(Banach in $\mathbb{R}^3$)

If $U_1, \ldots, U_n$ finite volume in $\mathbb{R}^n$, then there is a hyperplane that bisects each U_i



Polynomial Ham Sandwich

$U_1, \dots, U_N$ finite volume open sets in $\mathbb{R}^n$

If $N < \binom{D+n}{n}$, then there is a non-zero polynomial $P \in \mathrm{Poly}_D(\mathbb{R}^n)$ that bisects each U_i.

Say that P bisects a finite set S

if $|\{s \in S : P(s) > 0\}| = |\{s \in S : P(s) < 0\}|$

Polynomial Ham Sandwich thm also applies to finite sets U_i

Replace pts by δ-balls, $\delta \to 0$

<u>Rmk</u>: linear HST $\Rightarrow$ polynomial HST (at least for finite sets)

via Veronese embedding.

$$(x_1, \dots, x_n) \longmapsto \underbrace{(x_1, x_2, \dots, x_n, x_1^2, x_1 x_2 \dots}_{\text{monomials deg} \leq D}$$

polynomial surface

hyperplane.

Pf of polynomial partitioning thm.

- Find $P_1 \in \mathrm{Poly}_1(\mathbb{R}^n)$ bisecting X
- Find P_2, low deg., bisects X_+, X_-
- Find P_3 - - - - -

$$\begin{tikzcd}[column sep=small] & & & X \arrow[dll] \arrow[drr] & & & \\ & X_+ \arrow[dl] \arrow[dr] & & & & X_- \arrow[dl] \arrow[dr] & \\ X_{++} & & X_{+-} & & X_{-+} & & X_{--} \end{tikzcd}$$

$P = P_1 P_2 \cdots P_J$

Can choose $\deg P_j \lesssim C(n) 2^{j/n}$ by PHST

$\deg P \le C(n) \sum_{j=1}^{J} 2^{j/n} \lesssim 2^{J/n}$

Choose J s.t. $2^{J/n} \sim D$, get $\lesssim D^n$ open sets each with $\lesssim \frac{|X|}{2^J}$ pts

Pf of ST

Simple estimates: $I(\mathfrak{S}, \mathcal{L}) \le L + S^2$

$I(\mathfrak{S}, \mathcal{L}) \le S + L^2$

If $L > S^2$ or $S > L^2$, then simple estimate $\Rightarrow$ ST.

Assume: $S^{1/2} \le L \le S^2$

Apply poly part. poly P deg $\le D$

each cell contains $\lesssim \frac{S}{D^2}$ pts of $\mathfrak{S}$

$$\mathfrak{S} = \mathfrak{S}_{cell} \cup \mathfrak{S}_{alg}$$

$\mathfrak{S}_{cell} \leftarrow$ pts outside $Z(P)$

i.e. pts in cells.

$\mathfrak{S}_{cell} = \bigcup \mathfrak{S}_i \leftarrow$ pts of $\mathfrak{S}$ in ith cell.

$$\left|I(\mathfrak{S},\mathcal{L})\right| \leq \left|I(\mathfrak{S}_{cell},\mathcal{L})\right| + \left|I(\mathfrak{S}_{alg},\mathcal{L}_{cell})\right| + \left|I(\mathfrak{S}_{alg},\underbrace{\mathcal{L}_{alg}}_{\leq D.})\right|$$

simple estimate

$$\mathcal{L} = \mathcal{L}_{alg} \cup \mathcal{L}_{cell}$$

$\mathcal{L}_{alg}$: lines in $Z(P)$

$\mathcal{L}_{cell}$: other lines