Kakeya problem.

How "small" can a set in $\mathbb{R}^n$ be if it contains a line segment in every direction?

[Besicovitch] : can have measure zero

Conj must be n-dimensional

Finite field "toy model" [Wolff]

A set $K \subset \mathbb{F}_q^n$ is called a "Kakeya set" if it contains a line in every direction. What's the smallest possible size of K?

Thm (Dvir 2008)

If $K \subset \mathbb{F}_q^n$ is Kakeya set

$$|K| \geq c_n q^n.$$

Joints Problem

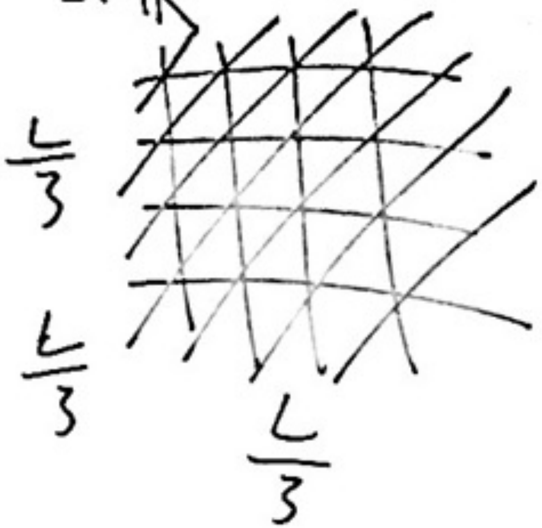
$\mathcal{L}$ - a set of lines in $\mathbb{R}^3$

a pt x a joint if it is incident to three non-coplanar lines in $\mathcal{L}$.

Q Given L lines, what's the max # of joints?

Non-example

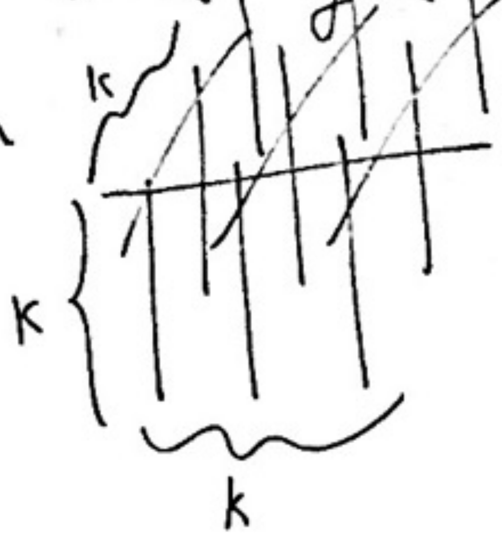
In $\mathbb{R}^2$



$\Theta(L^2)$ triple intersection points

not joints

In $\mathbb{R}^3$ grid



$K \approx \sqrt{\frac{L}{3}}$

$\#\text{joints} = k^3$

$= \Theta(L^{3/2})$

Thm (Guth-Katz 2008)

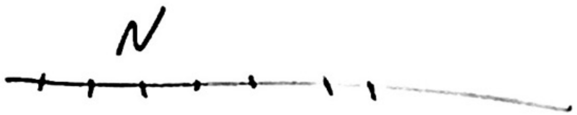
A set of L lines in $\mathbb{R}^3$ form $O(L^{3/2})$ joints.

Erdős distinct distances problem

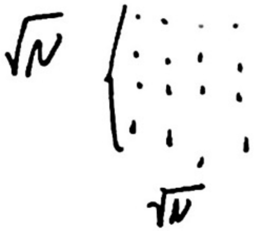
N points in $\mathbb{R}^2$

What's the min # of distinct pairwise distances that can occur?

E.g. N generic points $\binom{N}{2}$ distances



$N-1$ distances



$\Theta\left(\frac{N}{\sqrt{\log N}}\right)$ distances

Prior: $\gtrsim N^{0.86}$

Thm (Guth-Katz 2010)

distinct distances $\gtrsim \frac{N}{\log N}$

Notation: $f \lesssim g$ means $f = O(g)$

i.e. $\exists C > 0$ s.t. $|f| \leq Cg$

$f \gtrsim g$

$f \asymp g$ means $f \lesssim g$, $f \gtrsim g$

$f = \Theta(g)$

$(j, 2^j) \quad j = 1, \dots, 10^6$

Find a polynomial $P(X, Y)$ s.t.

$\underline{P(j, 2^j) = 0} \quad \forall j = 1, 2, \dots, 10^6$

$(X-1)(X-2)\cdots(X-10^6)$

Small degree?

<u>Claim</u> $\exists$ polynomial deg < 2000

$$P(X,Y) = \sum_{r+s<2000} a_{r,s} X^r Y^s$$

$\binom{2001}{2} > 10^6$ coefficients to choose

10^6 constraints.

<u>Parameter Counting</u>

$\mathbb{F}$ field

$\mathrm{Poly}_D(\mathbb{F}^n)$ space of polynomials in n variables total degree $\leq D$

Vec sp / $\mathbb{F}$

<u>Prop</u> Finite $S \subset \mathbb{F}^n$.

If $\underline{\dim \mathrm{Poly}_D(\mathbb{F}^n)} > |S|$

then $\exists$ nonzero $P \in \mathrm{Poly}_D(\mathbb{F}^n)$ vanishing on S.

What's the $\dim \mathrm{Poly}_D(\mathbb{F}^n)$

$$\dim \mathrm{Poly}_D(\mathbb{F}^n) = \binom{D+n}{n} \geq \frac{D^n}{n!} \geq \left(\frac{D}{n}\right)^n$$

$$x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n} \qquad d_1+d_2+\dots+d_n \leq D \qquad \left(d_i \leq \frac{D}{n}\right)$$

<u>Cor</u> $n \geq 2$, $S \subset \mathbb{F}^n$

$\exists$ a nonzero poly P deg $\leq n|S|^{1/n}$

vanishing on S.

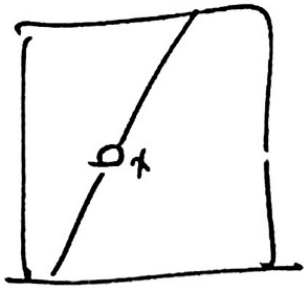
<u>Vanishing lemma</u>

<u>Lem</u> If $P \in \mathrm{Poly}_D(\mathbb{F})$ vanishes at $D+1$ points, then $P \equiv 0$

<u>Cor</u> If $P \in \mathrm{Poly}_D(\mathbb{F}^n)$ vanishes at $D+1$ points on some line ℓ, then it vanishes on all of ℓ.

Finite field Nikodym Problem.

$N \subset \mathbb{F}_q^n$ is a Nikodym set if $\forall x \in \mathbb{F}_q^n$, $\exists$ line $L(x) \ni x$ s.t. $L(x) \setminus \{x\} \subset N$



Thm (Dvir 2008) Any Nikodym set $N \subset \mathbb{F}_q^n$ has $|N| \geq c_n q^n$

$c_n = (10n)^{-n}$

Pf By contradiction, N is Nikodym set with $|N| < (10n)^{-n} q^n$

By parameter counting, $\exists P$ non-zero polynomial vanishing on N

s.t. $\deg P \leq n|N|^{1/n} \leq \frac{q}{10} < q-1$

Claim P vanishes at every pt in $\mathbb{F}_q^n$

Pf $x \in \mathbb{F}_q^n$. $\exists L(x) \setminus \{x\} \subset N$

P vanishes on N

vanishes at $\geq q-1$ pts on $L(x)$

By van. lem., $P(x) = 0$ //

Lem If $P \in \mathrm{Poly}_{q-1}(\mathbb{F}_q^n)$ vanishes at every point of $\mathbb{F}_q^n$, then $P \equiv 0$.

$$\geq \left(\frac{D}{n}\right)^n$$

$$\left(l_i \leq \frac{D}{n}\right)$$

Careful

$x^q - x$ in $\mathbb{F}_q$

nonzero
vanishes everywhere.

Pf Induction on n:

$n=1$ van. lemma ✓

$$P(x_1, \ldots, x_n) = \sum_{j=0}^{q-1} P_j(x_1, \ldots, x_{n-1}) x_n^j$$

...hes on all of l.

i.e. $\forall a \in \mathbb{F}_q^n \setminus \{0\}$

$\exists b \in \mathbb{F}_q^n$ s.t.

$\{at+b \mid t \in \mathbb{F}_q\} \subset K.$

Pf Suppose K is Kakeya set with $|K| < (10n)^{-n} q^n$.

$\exists$ nonzero $P \in \mathrm{Poly}_{q-2}(\mathbb{F}_q^n)$ vanishing on K.

Write $\underbrace{P = \underbrace{P_D}_{\text{homog poly deg}=D} + \underbrace{Q}_{\deg < D}}$

Let $a \in \mathbb{F}_q^n \setminus \{0\}$. Let $b \in \mathbb{F}_q^n$ s.t. $\{at+b \mid t \in \mathbb{F}_q\} \subset K$

$R(t) = P(at+b)$ vanishes $\forall t \in \mathbb{F}_q$

$\deg R < q \Rightarrow$ all coeff of R are zero

Coefficient of t^D in R is $P_D(a) = 0$. (a arbitrary)

$P(at+b) = P_D(at+b) + Q(at+b)$

$\therefore P_D(a) = 0 \quad \forall a \in \mathbb{F}_q^n \setminus \{0\}$

$P_D(0) = 0$

$\Rightarrow P_D \equiv 0$ contradiction $\square$

Alternative viewpoint

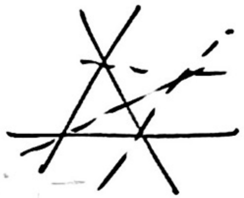
- If K small, $\exists$ poly P $\deg < q$ vanishing on K.
- Since P vanishes ~~at~~ on a line of every direction, P must vanishes on all points at infinity e.g. $\mathbb{P}\mathbb{F}_q^n \setminus \mathbb{F}_q^n$
- P vanishes at too many points.

Joints problem

$\mathcal{L}$ - set of lines in $\mathbb{R}^3$

joint - pt incident to 3 non-coplanar lines.

Ex 1  3-D grid

Ex 2 6 lines $\rightarrow$ 4 joints. 

S planes in general position

$L = \binom{S}{2}$ - lines from pairs of planes

$L^{3/2} \approx \binom{S}{3}$ triple intersections - joints

Thm (Guth-Katz, pf simplified by Kaplan-Sharir-Shustin/Quilodrán)

Any L lines in $\mathbb{R}^3$ determine $\leq 10 L^{3/2}$ joints.

Main lemma $\mathcal{L}$ lines in $\mathbb{R}^3$, J joints, then one of the lines contains $\leq 3J^{1/3}$ joints.

Pf of thm assuming lem

$J(L)$ = max # joints with L lines

$$J(L) \leq \underline{J(L-1)} + 3J(L)^{1/3}$$

$$\leq J(L-2) + 2 \cdot 3J(L)^{1/3}$$

$$J(L) \leq L \cdot 3J(L)^{1/3}$$

$$\Rightarrow J(L) \leq 10 L^{2/3}.$$

Let P be the minumum deg nonzero poly vanishing at every joint.

By parameter counting,

$$\deg P \leq 3J^{1/3}$$

If every line contains $> 3J^{1/3}$ joints, then P vanishes on all lines in $\mathcal{L}$.

Gradient of $F : \mathbb{R}^3 \to \mathbb{R}$.

$$\nabla F = \left(\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \frac{\partial F}{\partial x_3}\right)$$

<u>Lem</u> If x is joint in $\mathcal{L}$, and if a smooth function $F : \mathbb{R}^3 \to \mathbb{R}$ vanishes on lines in $\mathcal{L}$, then ∇F vanishes at x.

<u>Pf</u>. Let v_1, v_2, v_3 be the directional vectors of the 3 lines thru x.

Directional derivatives of F

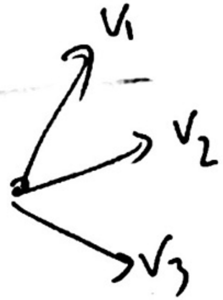
$$\nabla F(x) \cdot v_i = 0 \quad \forall i = 1, 2, 3.$$

But v_1, v_2, v_3 span $\mathbb{R}^3$.

$$\Rightarrow \nabla F(x) = 0$$

Thus ∇P vanishes at joints.

$$\nabla P = \left(\frac{\partial P}{\partial x_1}, \frac{\partial P}{\partial x_2}, \frac{\partial P}{\partial x_3}\right)$$



<u>By minimality of deg P</u>,

$\nabla P \equiv 0$. $\Rightarrow P = \text{const}$

$\Rightarrow$ no joints