Tiling: Coloring and Weights

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Main discussion: Packing boxes with bricks

Probably the most basic problem about tiling is to show that an 8×8 chessboard with two opposite corners removed cannot be tiled with dominoes. Many olympiad problems play on variations of this idea. (A related question worth considering: if one black square and one white square are removed from the chessboard, then can we always tile the rest with dominoes?) For example, consider the following generalization:

Problem 1: Let k be an integer. Which $m \times n$ boards can be tiled with $1 \times k$ tiles (rotations allowed)?

You could assign colors (labelled 1, 2, ...), or you could assign roots of unity. For $k = 3$ the coloring schemes are shown below (ω is a third root of unity).

1	2	3	1	2	3	$\cdots$
2	3	1	2	3	1	$\cdots$
3	1	2	3	1	2	$\cdots$
1	2	3	1	2	3	$\cdots$
2	3	1	2	3	1	$\cdots$
3	1	2	3	1	2	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

1	ω	ω^2	1	ω	ω^2	$\cdots$
ω	ω^2	1	ω	ω^2	1	$\cdots$
ω^2	1	ω	ω^2	1	ω	$\cdots$
1	ω	ω^2	1	ω	ω^2	$\cdots$
ω	ω^2	1	ω	ω^2	1	$\cdots$
ω^2	1	ω	ω^2	1	ω	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The idea is that each tile covers all three colors exactly once, so all the colors must appear the same number of times. In the roots of unity method, each tile covers a sum of zero since $1+\omega+\cdots+\omega^{k-1} = 0$, and the sum of the whole board must be zero. It's easy to check that this happens if and only if k divides m or n. (check this!) $\square$

In this problem, these two approaches amount to the same thing. Each method has its own advantages. In this discussion, we show how the roots of unity approach can be extended to other problems. Many problems given for practice at the end use the coloring approach extensively.

Consider the following continuous analogue of Problem 1. It is also a generalization, as it implies the Problem 1 (why?).

Problem 2: Show that if a rectangle can be tiled by smaller rectangles each of which has at least one integer side, then the tiled rectangle has at least one integer side.

There was a paper that contains 14 proofs of this result![1] How many can you find?

[1] Stan Wagon, *Fourteen Proofs of a Result About Tiling a Rectangle*, Amer. Math. Monthly, 94 (1987) 601–617

Let's see how we can use the idea in Problem 1. In Problem 1, we are essentially assigning the square with coordinates (i,j) with the value $\omega^{i+j} = e^{(i+j)\frac{2\pi i}{k}}$. How can we extend this to the continuous case? One way to do this is to construct the function $f : \mathbb{R}^2 \to \mathbb{C}$.

$$f(x,y) = e^{2\pi i(x+y)}.$$

This function assigns "weights" to the points of the plane, just as we assigned weights to the squares of the board in Problem 1. In Problem 1, the sum of the roots of unity covered by a single tile is 0. Here, note that the integral of f over any horizontal or vertical line segment with integer length is zero. This is the key insight.

So, let us place the large rectangle on the 2-D Cartesian coordinate, with its bottom left corner at the origin (assume that all rectangles are placed with its sides parallel to the axes). We can check that the integral of f over any rectangle with one integer dimension is zero. Thus, if a tiling is possible, then the integral of f over the entire large rectangle must be zero as well. You can check that this is possible only when one of the sides of the large rectangle has integer length as well. The technical details of this argument can be summarized in the following integration:

$$\iint\limits_{[a,b]\times[c,d]} f\, dA = \int_a^b \int_c^d e^{2\pi i(x+y)}\, dy dx = \int_a^b e^{2\pi i x}\, dx \int_c^d e^{2\pi i y}\, dy = \left(\frac{e^{2\pi i b} - e^{2\pi i a}}{2\pi i}\right)\left(\frac{e^{2\pi i d} - e^{2\pi i c}}{2\pi i}\right).$$

This is a perfectly valid solution. However, it uses calculus (gasp!). Moreover, it uses calculus with complex numbers! Can we get an elementary solution out of this? Well, let's see if we can at least reduce the solution to just calculus over the real numbers. Consider the following function:

$$f_1(x,y) = \sin(2\pi x)\sin(2\pi y)$$

The two weight functions f_1 and f share many common properties, and it turns that the solution still works if we had used f_1 instead of f. (However, the weight function $\cos(2\pi x)\cos(2\pi y)$ does not work. Why?)

We can simplify a bit more. Notice that we never really used anything about the exact curvature of sin, as we more or less only need the property that the integral of the function over $[x, x+1]$ is 0. So, why don't we "straighten" out our weight function, and use the following:

$$f_2(x,y) = g(x)g(y), \text{ where } g(x) = \begin{cases} 1 & \{x\} \le \frac{1}{2}, \\ -1 & \{x\} > \frac{1}{2}. \end{cases}$$

You can check that f_2 also does the trick! Moreover, f_3 allows us to come up with a combinatorial formulation of the solution—just consider the weight function as a black and white coloring of the board. You can work out the details yourself. □

Excellent! We just found three of the fourteen solutions.

Now, let's go back to the discrete case, but let's move up a dimension. Can you tile a $6 \times 6 \times 6$ with $1 \times 2 \times 4$ boxes? What can we say in general?

Problem 3: (de Bruijn) If the box $A_1 \times \cdots \times A_n$ can be tiled with bricks $a_1 \times \cdots \times a_n$, then show that for each i, a_i divides some A_j. (Note that this does not necessarily mean that the box is a multiple of the brick, e.g. the box $1 \times 5 \times 6$ and brick $1 \times 2 \times 3$.)

We say that a d-dimensional box A is a *multiple* of a d-dimensional box B if box A can be tiled by B in such a way that all the copies of B's are placed in the same orientation. Equivalently, A

is a multiple of B if there is some permutation σ on $\{1, 2, \ldots, n\}$ such that the i-th component of A is an integer multiple of the $\sigma(i)$-th component of B for each i.

Problem 3 isn't too difficult. In fact, it's pretty much the same as Problem 1. Let's focus on one particular a_i, and use the same idea as in problem 1 with $k = a_i$ to show that one of the dimensions of the box is divisible by a_i. □

Now, let's restrict ourselves to a certain special case:

Problem 4: (de Bruijn) Suppose that the brick $a_1 \times \cdots \times a_n$ satisfies the divisibility relations $a_1 \mid a_2 \mid \cdots \mid a_n$. Then the box $A_1 \times \cdots \times A_n$ can be tiled with the brick if and only if it's a multiple of the brick.

Let's work our way down through the a_i's one by one, starting from the largest. By Problem 3, there is some A_i that is divisible by a_n, say $a_n \mid A_n$. Now, dropping the last coordinate and take a cross-section of dimension $A_1 \times \cdots \times A_{n-1}$. If any tiles of this box has a side of length a_n, then the divisibility condition allows us to cut up the tiles into $a_1 \times \cdots \times a_{n-1}$ tiles. So now we end up with a tiling of the box $A_1 \times \cdots \times A_{n-1}$ with the bricks $a_1 \times \cdots \times a_{n-1}$. Repeat the argument. □

However, if the brick does not satisfy the chain of divisibilities, then there is always some box that can be tiled by the brick without being a multiple of the brick (exercise: prove this!).

Now, what if we are allowed to use two bricks, but with restricted orientations? Then, there is a definitive criterion for when we can tile a box with the bricks. Note that we don't even require the dimensions of the bricks to the be integers!

Problem 5: (Bower and Michael) Prove that the d-dimensional box R can be tiled by translates of two given d-dimensional bricks B_1 and B_2 if and only if R can be partitioned by a hyperplane into two sub-boxes R_1 and R_2 such that R_i is a multiple of B_i for $i = 1, 2$.

Note that the not *every* tiling has to be bipartite, but the existence of a non-bipartite tiling implies the existence of a bipartite tiling.

The most difficult part of the problem is the 2-D case. However, we have done most of the work for that already! (Where?) The rest is left as exercise.

Problems

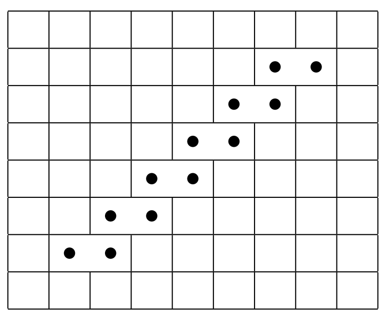
Key ideas: color the board in some way that gives some constraints or invariants. Often, this involves marking a certain subset of the board. Looking at parities and other modulos is a good idea. Sometimes you may have to use more than one coloring schemes simultaneously[2] to solve a problem. Finally, don't forget that some problem have two (related) parts—proving a constraint and constructing an example.

1. Is there a closed knight's tour on a 5×5 chessboard?

2. For which n is there a closed knight's tour on a $4 \times n$ chessboard?

3. (Tournament of Towns 2004) Given two rectangles A and B, such that one can tile a rectangle similar to B using copies of A, show that one can tile a rectangle similar to A using copies of B.

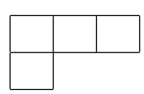
4. (Canada 2007) What is the maximum number of dominoes which can be placed on an 8×9 board if six are already placed as shown below?



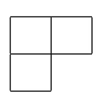
5. What is the smallest number of squares on an 8×8 chessboard which would have to be painted so that no 3×1 rectangle could be placed on the board without covering a painted square.

6. Which single square can be removed from a 7×7 board so that the rest can be tiled with 1×3 trominos.

7. Prove that a 4×11 rectangle cannot be tiled by L-shaped tetrominoes.



8. (Russia 1996) Can a 5×7 board be covered by L-trominos, not crossing its boundary, in several layers, so that each square of the board is covered by the same number of trominos?



[2]You may have seen certain "cool" coloring proofs that use the Klein four-group. Those might not actually be as fancy as you thought. The Klein four-group is the group with four elements $\{e, a, b, c\}$ satisfying the relations $a^2 = b^2 = c^2 = abc = e$, but in fact, it is isomorphic to the group of two-dimensional coordinates in mod 2 under addition, i.e., $\{(0,0), (1,0), (0,1), (1,1)\}$. Thus, any coloring proof that uses the Klein four-group can be replicated by multiple applications of a black-white coloring scheme.

9. (Russia 2002) A rectangle is partitioned into 100 L-trominos and some 1×3 tiles. Someone chosen chosen 96 of these L-trominoes and merge them in pairs into 48 2×3 rectangles. Prove that one can translate the remaining 4 L-trominoes and merge them into two 2×3 rectangles.

10. A 6×6 board is tiled by dominoes. Show that there is a line that cuts the board into two parts without cutting any domino.

11. (Iurie Boreico) Let n be a positive integer such that $\gcd(n, 6) = 1$, and let k and l be positive integers. The entries of a $k \times l$ table are all $+1$ or -1. One can simultaneously change the signs of any n consecutive entries horizontally, vertically, or diagonally. Prove that one can eventually make all the entires negative numbers if and only if n divides k or n divides l.

12. (Tournament of Towns 1993) On a 10×10 square board we are trying we place ten rectangles: one 1×4, two 1×3, three 1×2 and four 1×1. Prove that if we arbitrarily place the rectangles on the board but in the aforementioned order, then at each step, it is always possible to fit the rectangle into the board so that it does not share a point (even on the boundaries) with an existing rectangle.

13. There is a 5×5 array of lights, such that at each step, we may toggle all the lights in any 2×2, 3×3, 4×4 or 5×5 sub-square. Initially all the lights are switched off. After a certain number of toggles, exactly one light is switched on. Find all the possible positions of the light.

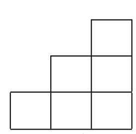
14. (APMO 2007) A regular (5×5)-array of lights is defective, so that toggling the switch for one light causes each adjacent light in the same row and in the same column as well as the light itself to change state, from on to off, or from off to on. Initially all the lights are switched off. After a certain number of toggles, exactly one light is switched on. Find all the possible positions of the light.

15. (USAMO 1998) A computer screen shows a 98×98 chessboard, colored in the usual way. One can select with a mouse any rectangle with sides on the lines of the chessboard and click the mouse button: as a result, the colors in the selected rectangle switch (black becomes white, white becomes black). Determine the minimum number of mouse clicks needed to make the chessboard all one color.

16. (Balkan 2000) Determine the maximum number of $1 \times 10\sqrt{2}$ rectangles that can be placed on a 50×90 rectangle without overlap and so that the small rectangles have its sides parallel to the large rectangle.

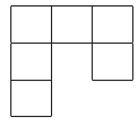
17. (IMO Shortlist 2002) For n an odd positive integer, the unit squares of an $n \times n$ chessboard are colored alternately black and white, with the four corners colored black. For which values of is it possible to cover all the black squares with non-overlapping L-trominos? When it is possible, what is the minimum number of L-trominos needed?

18. (IMO Shortlist 2000) A staircase-brick with 3 steps of width 2 is made of 12 unit cubes. Determine all integers for which it is possible to build a cube of side n using such bricks.



(The staircase-brick is made up of two such layers.)

19. (IMO 2004) Define a "hook" to be a figure made up of six unit squares as shown below in the picture, or any of the figures obtained by applying rotations and reflections to this figure.



Determine all $m \times n$ rectangles that can be tiled with hooks? (i.e. no gaps, no overlaps, and no part of a hook lies outside the rectangle).[3]

Fun facts about tiling

- (Fish and Temperly 1961; Kasterleyn 1961) The number of tilings of a $2m \times 2n$ rectangle with $2mn$ dominoes is equal to

$$4^{mn} \prod_{j=1}^{m} \prod_{k=1}^{n} \left(\cos^2 \frac{j\pi}{2m+1} + \cos^2 \frac{k\pi}{2n+1} \right)$$

  This is a remarkable formula. It's not even clear that the product is an integer!

- (Laczkovich and Szekeres 1995) For which x can a square be tiled with finitely many rectangles similar to a $1 \times x$ rectangle (in any orientation)?

  The answer is that this is possible if and only if x is the root of a polynomial with integer coefficients, and all the roots of the minimal polynomial of x has positive real part.

  For example, we cannot tile a square with rectangles similar to $1 \times \sqrt{2}$, but we can tile a square with rectangles similar to $1 \times \left(\sqrt{2} + \frac{17}{12}\right)$.

[3]For an analysis of a large number of polyomino tiles in terms of which rectangles each can tile, see `http://www.math.ucf.edu/~reid/Polyomino/rectifiable_data.html`. This database contains the analysis of the "hook" polyomino way before the 2004 IMO. Consequently, the problem received some complaints on MathLinks after the contest because it was "well-known." Nevertheless, only 11 students solved the problem at the IMO, and no one got 6 points (only one contestant got a 5 . . . and there's an interesting story behind that . . .).