# Similarity

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1. Let $ABCD$ be a convex quadrilateral. Show that

$$AC^2 \cdot BD^2 = AB^2 \cdot CD^2 + AD^2 \cdot BC^2 - 2AB \cdot BC \cdot CD \cdot DA \cos(A + C).$$

2. (IMO Shortlist 1998) Let $ABCDEF$ be a convex hexagon such that $\angle B + \angle D + \angle F = 360^\circ$ and

$$\frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} = 1.$$

   Prove that

$$\frac{BC}{CA} \cdot \frac{AE}{EF} \cdot \frac{FD}{DB} = 1.$$

3. A circle $\omega$ is inscribed in a quadrilateral $ABCD$. Let $I$ be the center of $\omega$. Show that

$$BI^2 + \frac{AI \cdot BI \cdot CI}{DI} = AB \cdot BC.$$

4. (Turkey 1998) Let $ABC$ be a triangle. Suppose that the circle through $C$ tangent to $AB$ at $A$ and the circle through $B$ tangent to $AC$ at $A$ have different radii, and let $D$ be their second intersection. Let $E$ be the point on the ray $AB$ such that $AB = BE$. Let $F$ be the second intersection of the rat $CA$ with the circle through $A, D, E$. Prove that $AF = AC$.

5. A circle with center $O$ passes through the vertices $A$ and $C$ of triangle $ABC$ and intersects segments $AB$ and $BC$ again at distinct points $K$ and $N$, respectively. The circumcircles of triangles $ABC$ and $KBN$ intersects at exactly two distinct points $B$ and $M$. Prove that $\angle OMB = 90^\circ$.

6. Circles $\omega_1$ and $\omega_2$ meet at points $O$ and $M$. Circle $\omega$, centered at $O$, meet circles $\omega_1$ and $\omega_2$ in four distinct points $A, B, C$ and $D$, such that $ABCD$ is a convex quadrilateral. Lines $AB$ and $CD$ meet at $N_1$. Lines $AD$ and $BC$ meet at $N_2$. Prove that $N_1N_2 \perp MO$.

7. (Crux) Points $O$ and $H$ are the circumcenter and orthocenter of acute triangle $ABC$, respectively. The perpendicular bisector of segment $AH$ meets sides $AB$ and $AC$ at $D$ and $E$, respectively. Prove that $\angle DOA = \angle EOA$.

8. (IMO Shortlist 2000) Let $ABCD$ be a convex quadrilateral with $AB$ not parallel to $CD$, and let $X$ be a point inside $ABCD$ such that $\angle ADX = \angle BCX < 90^\circ$ and $\angle DAX = \angle CBX < 90$. If the perpendicular bisectors of segments $AB$ and $CD$ intersect at $Y$, prove that $\angle AYB = 2\angle ADX$.