Power of a Point

Solutions

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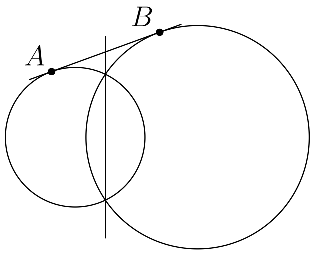
Practice problems:

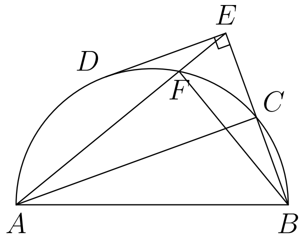
1. Let Γ_1 and Γ_2 be two intersecting circles. Let a common tangent to Γ_1 and Γ_2 touch Γ_1 at A and Γ_2 at B. Show that the common chord of Γ_1 and Γ_2, when extended, bisects segment AB.



Solution. Let the common chord extended meet AB at M. Since M lies on the radical axis of Γ_1 and Γ_2, it has equal powers with respect to the two circles, so $MA^2 = MB^2$. Hence $MA = MB$.

2. Let C be a point on a semicircle of diameter AB and let D be the midpoint of arc AC. Let E be the projection of D onto the line BC and F the intersection of line AE with the semicircle. Prove that BF bisects the line segment DE.

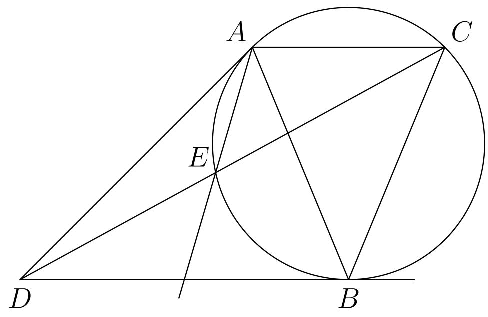
Solution.



Let Γ denote the circle with diameter AB, and Γ_1 denote the circle with diameter BE. Since $\angle AFB = 90°$, Γ_1 passes through F. Also since $\angle DEB = 90°$, Γ_1 is tangent to DE. From Problem 1, we deduce that the common chord BF of Γ and Γ_1 bisects their common tangent DE.

3. Let A, B, C be three points on a circle Γ with $AB = BC$. Let the tangents at A and B meet at D. Let DC meet Γ again at E. Prove that the line AE bisects segment BD.

Solution.



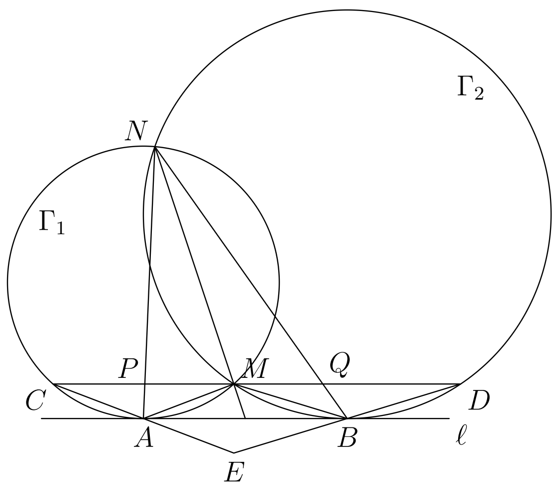
Let Γ_1 denote the circumcircle of ADE. By Problem 1 it suffices to show that Γ_1 is tangent to DB. Indeed, we have

$$\angle ADB = 180^\circ - 2\angle ABD = \angle ABC = \angle AEC,$$

which implies that Γ_1 is tangent to D.

4. (IMO 2000) Two circles Γ_1 and Γ_2 intersect at M and N. Let ℓ be the common tangent to Γ_1 and Γ_2 so that M is closer to ℓ than N is. Let ℓ touch Γ_1 at A and Γ_2 at B. Let the line through M parallel to ℓ meet the circle Γ_1 again at C and the circle Γ_2 again at D. Lines CA and DB meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.

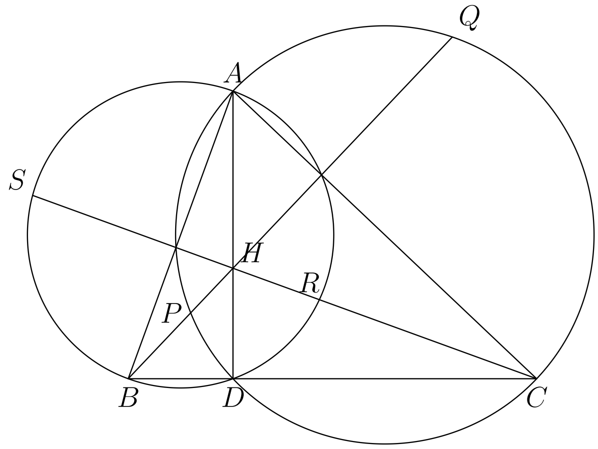
   Solution.



Extend NM to meet AB at X. Then by Problem 1, X is the midpoint of AB. Since PQ is parallel to AB, it follows that M is the midpoint of PQ. Since $\angle MAB = \angle MCE = \angle BAE$ and $\angle MBA = \angle MDE = \angle ABE$, we see that E is the reflection of M across AB. So EM the perpendicular bisector of PQ, and hence $EP = EQ$.

5. Let ABC be an acute triangle. Let the line through B perpendicular to AC meet the circle with diameter AC at points P and Q, and let the line through C perpendicular to AB meet the circle with diameter AB at points R and S. Prove that P, Q, R, S are concyclic.

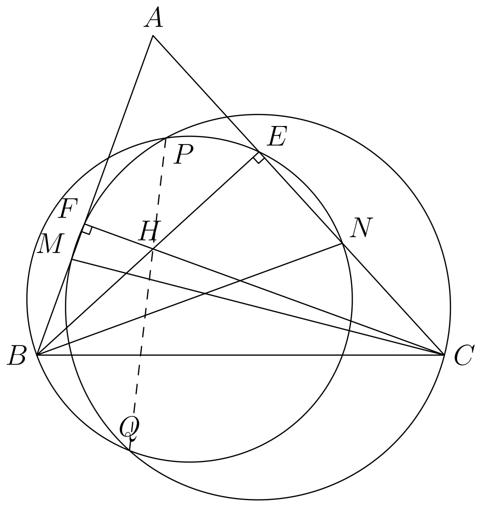
   Solution.



Let D be the foot of the perpendicular from A to BC, and let H be the orthocenter of ABC. Since $\angle ADB = 90^\circ$, the circle with diameter AB passes through D, so $HS \cdot HR = HA \cdot HD$ by power of a point. Similarly the circle with diameter AC passes through D as well, so $HP \cdot HQ = HA \cdot HD$ as well. Hence $HP \cdot HQ = HR \cdot HS$, and therefore by the converse of power of a point, P, Q, R, S are concyclic.

6. Let ABC be an acute triangle with orthocenter H. The points M and N are taken on the sides AB and AC, respectively. The circles with diameters BN and CM intersect at points P and Q. Prove that P, Q, and H are collinear.

   Solution.



   We want to show that H lies on the radical axis of the two circles, so it suffices to show that H has equal powers with respect to the two circles.

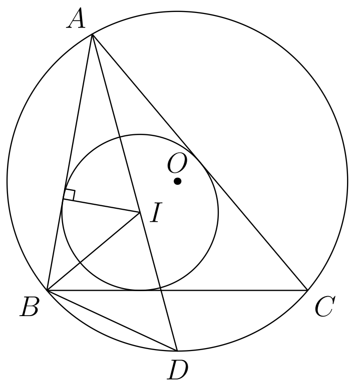
   Let BE and CF be two altitudes of ABC. Since $\angle BEN = 90^\circ$, E lies the circle with diameter BN. Hence the power of H with respect to the circle with diameter BN is $HB \cdot HE$. Similarly, the power of H with respect to the the circle with diameter CM is $HC \cdot HF$.

   Since $\angle BEC = \angle BFC = 90^\circ$, B, C, E, F are concyclic, hence $HB \cdot HE = HC \cdot HF$ by power of a point. It follows that H has equal powers with respect to the two circles with diameter AB and BC.

7. (Euler's relation) In a triangle with circumcenter O, incenter I, circumradius R, and inradius r, prove that

$$OI^2 = R(R - 2r).$$

Solution.



Let AI extended meet the circumcircle again at D. The power of I with respect to the circumcircle is equal to

$$-IA \cdot ID = IO^2 - R^2.$$

Let us compute the lengths of IA and ID. By consider the right triangle with one vertex A and the opposite side the radius of the incircle perpendicular to AB, we find $IA = r \sin \frac{A}{2}$. We have

$$\angle BID = \angle BAD + \angle ABI = \angle DAC + \angle IBC = \angle DBC + \angle IBC = \angle IBD.$$

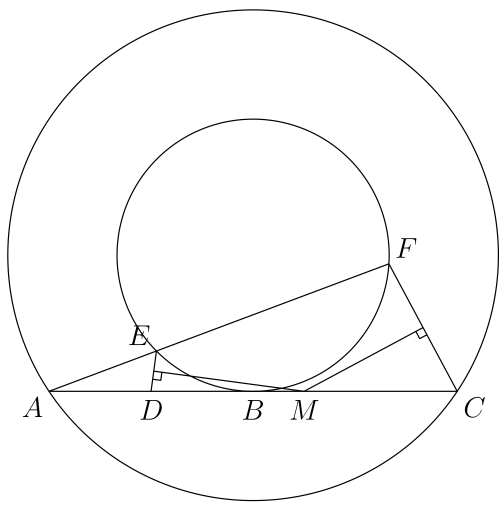
Thus $ID = BD = \frac{2R}{\sin \frac{A}{2}}$, where the last equality follows from the law of sines on triangle ABD. Hence

$$R^2 - IO^2 = IA \cdot ID = r \sin \frac{A}{2} \cdot \frac{2R}{\sin \frac{A}{2}} = 2Rr.$$

The result follows.

8. (USAMO 1998) Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be concentric circles, with $\mathcal{C}_2$ in the interior of $\mathcal{C}_1$. Let A be a point on $\mathcal{C}_1$ and B a point on $\mathcal{C}_2$ such that AB is tangent to $\mathcal{C}_2$. Let C be the second point of intersection of AB and $\mathcal{C}_1$, and let D be the midpoint of AB. A line passing through A intersects $\mathcal{C}_2$ at E and F in such a way that the perpendicular bisectors of DE and CF intersect at a point M on AB. Find, with proof, the ratio AM/MC.

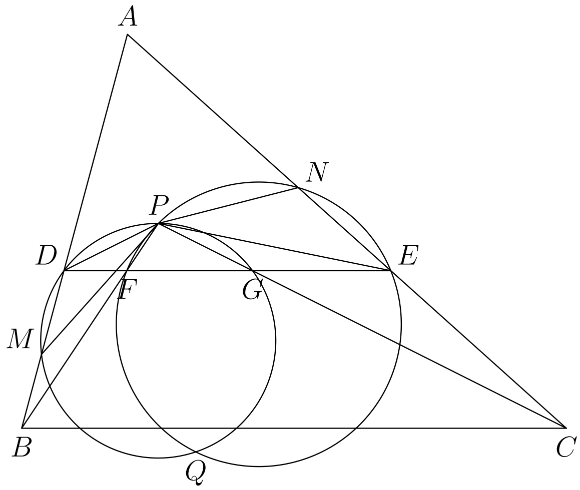
Solution.



Using power of point, we have $AE \cdot AF = AB^2 = AD \cdot AC$. Therefore, D, C, F, E are concyclic. The intersection M of the perpendicular bisectors of DE and CF must meet at the center of the circumcircle of $DCFE$. Since M is on DC, it follows that DC is the diameter of this circle. Hence M is the midpoint of DC. So $\frac{MC}{AC} = \frac{1}{2}\frac{DC}{AC} = \frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$. Thus $\frac{AM}{MC} = \frac{3}{5}$.

9. Let ABC be a triangle and let D and E be points on the sides AB and AC, respectively, such that DE is parallel to BC. Let P be any point interior to triangle ADE, and let F and G be the intersections of DE with the lines BP and CP, respectively. Let Q be the second intersection point of the circumcircles of triangles PDG and PFE. Prove that the points A, P, and Q are collinear.

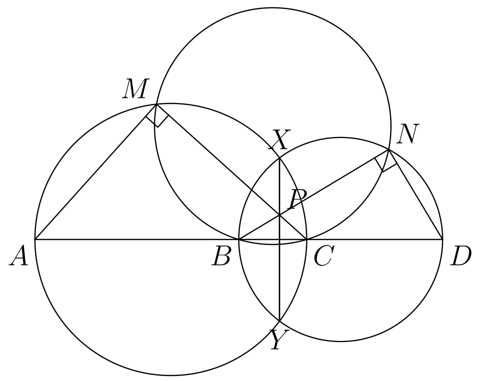
Solution.



Let the circumcircle of DPG meet line AB again at M, and let the circumcircle of EPF meet line AC again at N. Assume the configuration where M and N lie on sides AB and AC respectively (the arguments for the other cases are similar). We have $\angle ABC = \angle ADG = 180° - \angle BDG = 180° - \angle MPC$, so $BMPC$ is cyclic. Similarly, $BPNC$ is cyclic as well. So $BCNPM$ is cyclic. Hence $\angle ANM = \angle ABC = \angle ADE$, so M, N, D, E are concyclic. By power of a point, $AD \cdot AM = AE \cdot AD$. Therefore, A has equal power with respect to the circumcircles of DPG and the EPF, and thus A lies on line PQ, the radical axis.

10. (IMO 1995) Let A, B, C, and D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y . The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, and XY are concurrent.

Solution.



By power of a point, we have $PM \cdot PC = PX \cdot PY = PN \cdot PB$, so B, C, M, N are concyclic. Note that $\angle AMC = \angle BND = 90^\circ$ since they are subtended by diameters AC ad BD, respectively. Hence $\angle MND = 90^\circ + \angle MNB = 90^\circ + \angle MCA = 180^\circ - \angle MAD$. Therefore A, D, N, M are concyclic. Since AM, DN, XY are the three radical axes for the circumcircles of $AMXC$, $BXND$, and $AMND$, they concur at the radical center.