Power of a Point

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Power of a point is a frequently used tool in Olympiad geometry.

Theorem 1 (Power of a point)**.** *Let* Γ *be a circle, and* P *a point. Let a line through* P *meet* Γ *at points* A *and* B*, and let another line through* P *meet* Γ *at points* C *and* D*. Then*

$$PA \cdot PB = PC \cdot PD.$$

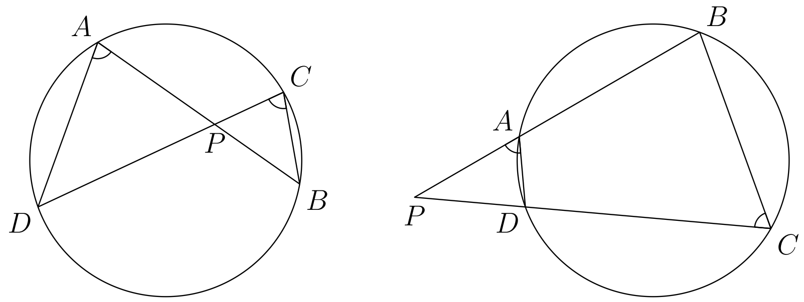


Figure 1: Power of a point.

Proof. There are two configurations to consider, depending on whether P lies inside the circle or outside the circle. In the case when P lies inside the circle, as the left diagram in Figure 1, we have $\angle PAD = \angle PCB$ and $\angle APD = \angle CPB$, so that triangles PAD and PCB are similar (note the order of the vertices). Hence $\frac{PA}{PD} = \frac{PC}{PB}$. Rearranging we get $PA \cdot PB = PC \cdot PD$.

When P lies outside the circle, as in the right diagram in Figure 1, we have $\angle PAD = \angle PCB$ and $\angle APD = \angle CPB$, so again triangles PAD and PCB are similar. We get the same result in this case. □

As a special case, when P lies outside the circle, and PC is a tangent, as in Figure 2, we have

$$PA \cdot PB = PC^2.$$

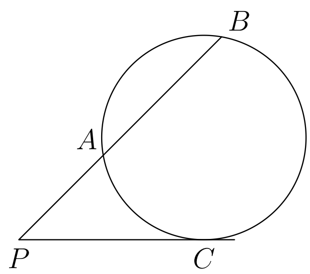


Figure 2: $PA \cdot PB = PC^2$

The theorem has a useful converse for proving that four points are concyclic.

Theorem 2 (Converse to power of a point)**.** *Let A, B, C, D be four distinct points. Let lines AB and CD intersect at P. Assume that either (1) P lies on both line segments AB and CD, or (2) P lies on neither line segments. Then A, B, C, D are concyclic if and only if $PA \cdot PB = PC \cdot PD$.*

Proof. The expression $PA \cdot PB = PC \cdot PD$ can be rearranged as $\frac{PA}{PD} = \frac{PC}{PB}$. In both configurations described in the statement of the theorem, we have $\angle APD = \angle CPB$. It follows by angles and ratios that triangles APD and CPB are similar (with the vertices in that order). Thus $\angle PAD = \angle PCB$. In both cases this implies that A, B, C, D are concyclic. □

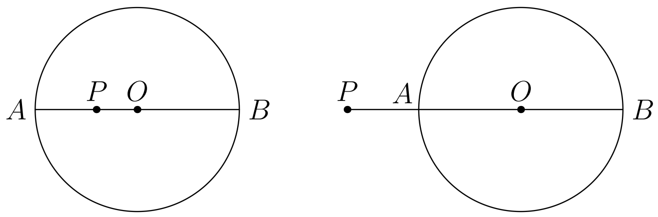
Suppose that Γ has center O and radius r. We say that the **power** of point P with respect to Γ is

$$PO^2 - r^2.$$

Let line PO meet Γ at points A and B, so that AB is a diameter. We'll use *directed lengths*, meaning that for collinear points P, A, B, an expression like $PA \cdot PB$ is a assigned a positive value if PA and PB point in the same direction, and a negative value if they point in opposite directions. Then

$$PA \cdot PB = (PO + OA)(PO + OB) = (PO - r)(PO + r) = PO^2 - r^2,$$

which is the power of P. So the power of a point theorem says that the this quantity equals to $PC \cdot PD$, where C and D are the intersection with Γ of any line through P.



By convention, the power of P is negative when P is inside the circle, and positive when P is outside the circle. When P is outside the circle, the power equals to the square of the length of the tangent from P to the circle.

Let Γ_1 and Γ_2 be two circles with different centers O_1 and O_2, and radii r_1 and r_2 respectively. The **radical axis** of Γ_1 and Γ_2 is the set of points whose powers with respect to Γ_1 and Γ_2 are equal. So it is the set of points P such that

$$PO_1^2 - r_1^2 = PO_2^2 - r_2^2.$$

The set of such points P form a line (one way to see this is write out the above relation using cartesian coordinates). By symmetry, the radical axis must be perpendicular to O_1 and O_2.

When Γ_1 and Γ_2 intersect, the intersection points A and B both have a power of 0 with respect to either circle, so A and B must lie on the radical axis. This shows that the radical axis coincide with the common chord when the circles intersect.

Sometimes when we need to show that some point lies on the radical axis or the common chord, it might be a good idea to show that the point has equal powers with respect to the two circles.

It is often helpful to examine the radical axes of more than just two circles.

Theorem 3 (Radical axis theorem)**.** *Given three circles, no two concentric, the three pairwise radical axes are either concurrent or all parallel.*

The common point of intersection of the three radical axes is known as the **radical center**.

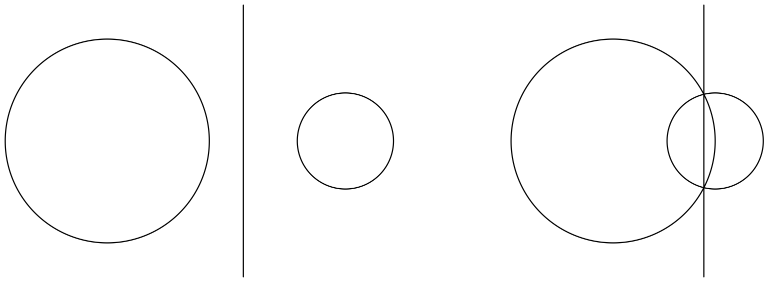


Figure 3: Radical axis

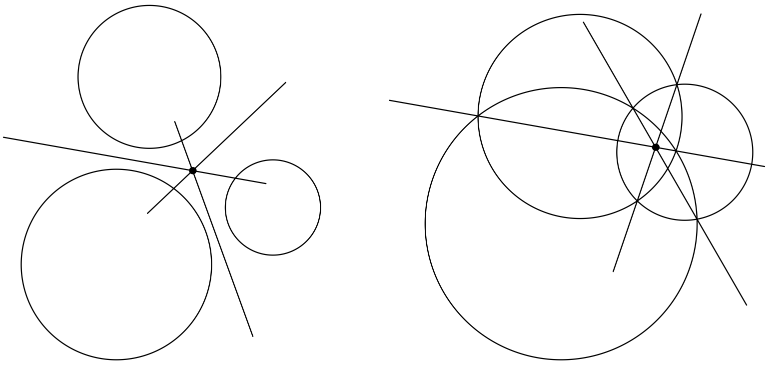
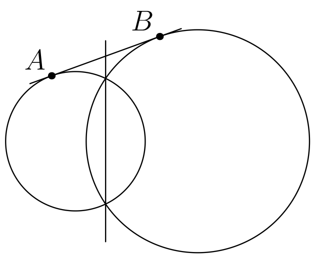


Figure 4: Radical center

Proof. Denote the three circles by Γ_1, Γ_2, and Γ_2, and denote the radical axes of Γ_i and Γ_j by ℓ_{ij}. Suppose that the radical axes are not all parallel. Let ℓ_{12} and ℓ_{13} meet at X. Since X lies on ℓ_{12}, it has equal powers with respect to Γ_1 and Γ_2. Since X lies on ℓ_{13}, it has equal powers with respect to Γ_1 and Γ_3. Therefore, X has equal powers with respect to all three circles, and hence it must lie on ℓ_{23} as well. □

Practice problems:

1. Let Γ_1 and Γ_2 be two intersecting circles. Let a common tangent to Γ_1 and Γ_2 touch Γ_1 at A and Γ_2 at B. Show that the common chord of Γ_1 and Γ_2, when extended, bisects segment AB.



2. Let C be a point on a semicircle of diameter AB and let D be the midpoint of arc AC. Let E be the projection of D onto the line BC and F the intersection of line AE with the semicircle. Prove that BF bisects the line segment DE.

3. Let A, B, C be three points on a circle Γ with $AB = BC$. Let the tangents at A and B meet at D. Let DC meet Γ again at E. Prove that the line AE bisects segment BD.

4. (IMO 2000) Two circles Γ_1 and Γ_2 intersect at M and N. Let ℓ be the common tangent to Γ_1 and Γ_2 so that M is closer to ℓ than N is. Let ℓ touch Γ_1 at A and Γ_2 at B. Let the line through M parallel to ℓ meet the circle Γ_1 again at C and the circle Γ_2 again at D. Lines CA and DB meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.

5. Let ABC be an acute triangle. Let the line through B perpendicular to AC meet the circle with diameter AC at points P and Q, and let the line through C perpendicular to AB meet the circle with diameter AB at points R and S. Prove that P, Q, R, S are concyclic.

6. (Euler's relation) In a triangle with circumcenter O, incenter I, circumradius R, and inradius r, prove that
$$OI^2 = R(R - 2r).$$

7. (USAMO 1998) Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be concentric circles, with $\mathcal{C}_2$ in the interior of $\mathcal{C}_1$. Let A be a point on $\mathcal{C}_1$ and B a point on $\mathcal{C}_2$ such that AB is tangent to $\mathcal{C}_2$. Let C be the second point of intersection of AB and $\mathcal{C}_1$, and let D be the midpoint of AB. A line passing through A intersects $\mathcal{C}_2$ at E and F in such a way that the perpendicular bisectors of DE and CF intersect at a point M on AB. Find, with proof, the ratio AM/MC.

8. Let ABC be a triangle and let D and E be points on the sides AB and AC, respectively, such that DE is parallel to BC. Let P be any point interior to triangle ADE, and let F and G be the intersections of DE with the lines BP and CP, respectively. Let Q be the second intersection point of the circumcircles of triangles PDG and PFE. Prove that the points A, P, and Q are collinear.

9. (IMO 1995) Let A, B, C, and D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y . The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, and XY are concurrent.