Cyclic Quadrilaterals — The Big Picture

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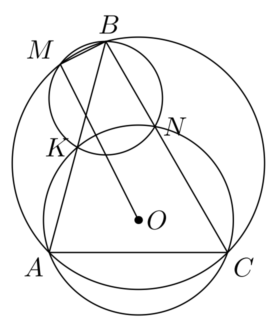
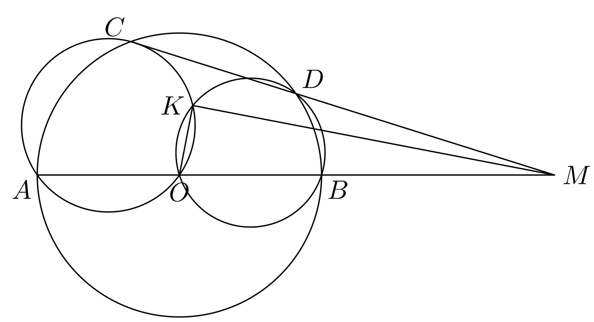
An important skill of an olympiad geometer is being able to recognize known configurations. Indeed, many geometry problems are built on a few common themes. In this lecture, we will explore one such configuration.

1 What Do These Problems Have in Common?

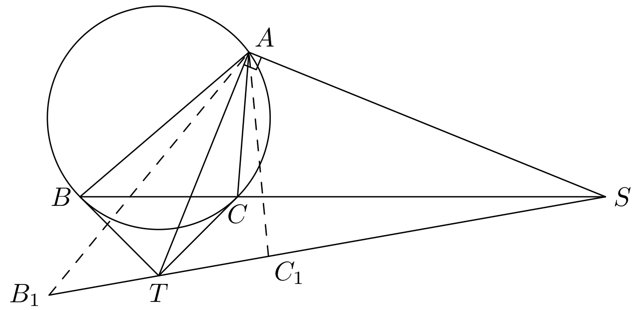
1. (IMO 1985) A circle with center O passes through the vertices A and C of triangle ABC and intersects segments AB and BC again at distinct points K and N, respectively. The circumcircles of triangles ABC and KBN intersects at exactly two distinct points B and M. Prove that $\angle OMB = 90°$.



2. (Russia 1995; Romanian TST 1996; Iran 1997) Consider a circle with diameter AB and center O, and let C and D be two points on this circle. The line CD meets the line AB at a point M satisfying $MB < MA$ and $MD < MC$. Let K be the point of intersection (different from O) of the circumcircles of triangles AOC and DOB. Show that $\angle MKO = 90°$.



3. (USA TST 2007) Triangle ABC is inscribed in circle ω. The tangent lines to ω at B and C meet at T. Point S lies on ray BC such that $AS \perp AT$. Points B_1 and C_1 lies on ray ST (with C_1 in between B_1 and S) such that $B_1T = BT = C_1T$. Prove that triangles ABC and AB_1C_1 are similar to each other.



Although these geometric configurations may seem very different at first sight, they are actually very related. In fact, they are all just bits and pieces of one big diagram!

2 One Big Diagram

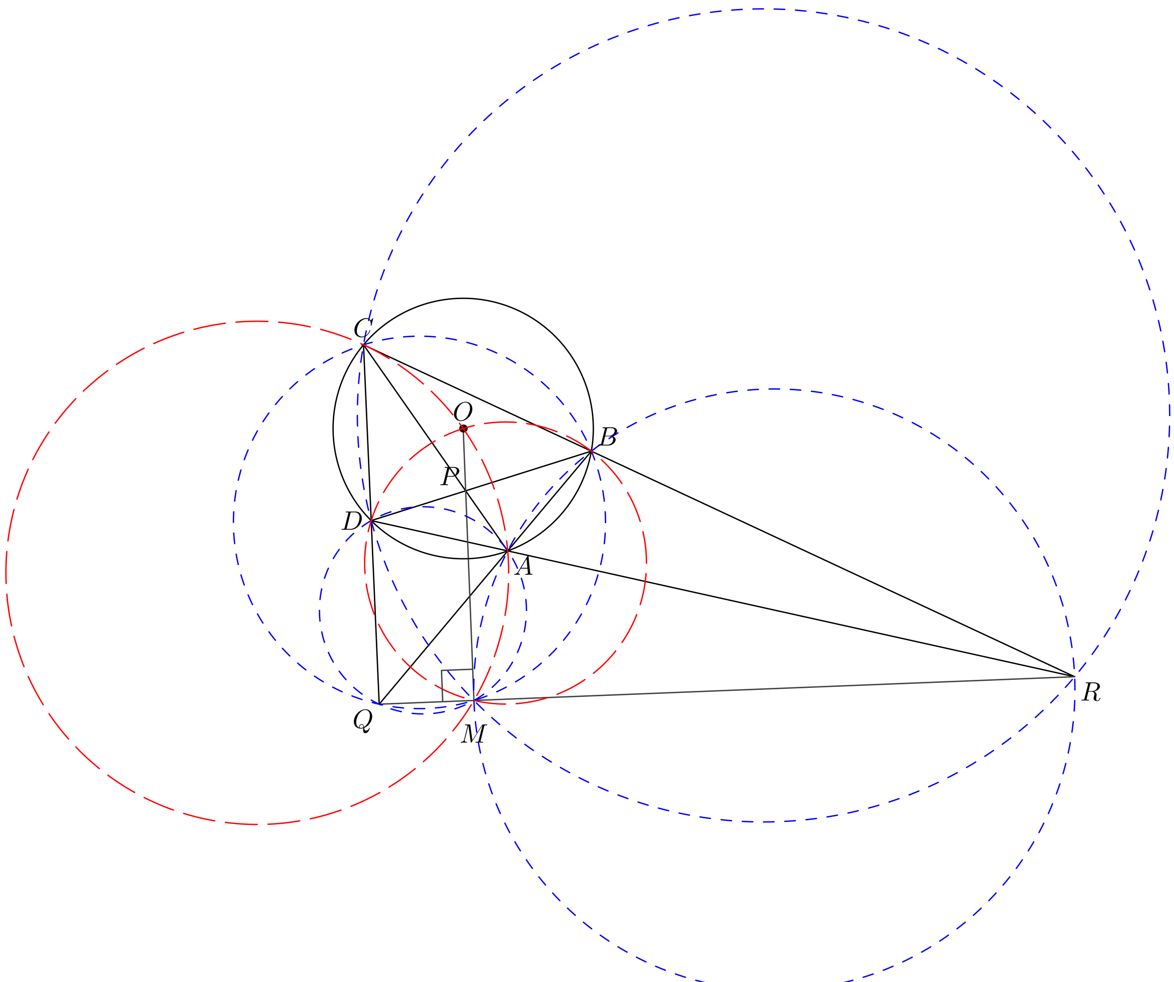


Figure 1: The big picture.

In this lecture, we will try to understand the features of Figure 1. There are a lot of things going on in this diagram, and it can be frightening to look at. Don't worry, we will go through it bits and pieces at time. In the process, we will discuss some geometric techniques that are useful in other places as well.

(Can you tell where to find each of the problems in Section 1 in Figure 1? You probably can't at this point, but hopefully you will be able to by the end of this lecture.)

3 Miquel's Theorem and Miquel Point

Fact 1 (Miquel's Theorem)**.** Let ABC be a triangle, and let X, Y, Z be points on lines BC, CA, AB, respectively. Assume that the six points A, B, C, X, Y, Z are all distinct. Then the circumcircles of AYZ, BZX, CXY pass through a common point.

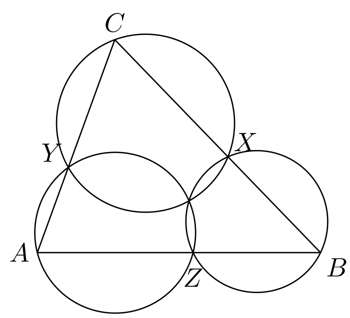


Figure 2: Diagram for Fact 1 (Miquel's Theorem).

Exercise 1. Prove Fact 1. (This is very easy. Just chase[1] a few angles.)

Fact 2 (Miquel point)**.** Let $\ell_1, \ell_2, \ell_3, \ell_4$ be four lines in the plane, no two parallel. Let $\mathcal{C}_{ijk}$ denote the circumcircle of the triangle formed by the lines ℓ_i, ℓ_j, ℓ_k (these circles are called *Miquel circles*). Then $\mathcal{C}_{123}, \mathcal{C}_{124}, \mathcal{C}_{134}, \mathcal{C}_{234}$ pass through a common point (called the *Miquel point*).

Exercise 2. Prove Fact 2. (Hint: apply Theorem 1)

We want to specialize to the case of a cyclic quadrilateral.

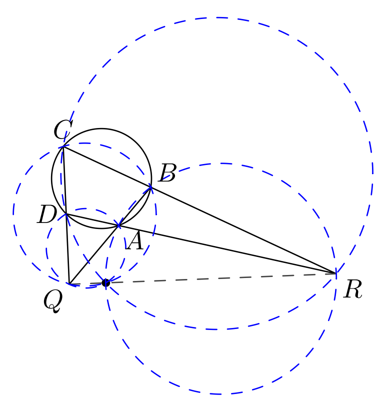


Figure 3: Miquel point for a cyclic quadrilateral $ABCD$.

[1]If you are bothered by configuration and orientation issues (and you should be!), use directed angles.

Fact 3. Let $ABCD$ be a quadrilateral. Let lines AB and CD meet at Q, and lines DA and CB meet at R. Then the Miquel point of $ABCD$ (i.e., the second intersection point of the circumcircles of ADQ and ABR) lies on the line QR if and only if $ABCD$ is cyclic.

Exercise 3. Prove Fact 3. (This is again just easy angle chasing.)

4 An Important Result about Spiral Similarities

A *spiral similarity*[2] about a point O (known as the center of the spiral similarity) is a composition of a rotation and a dilation, both centered at O.

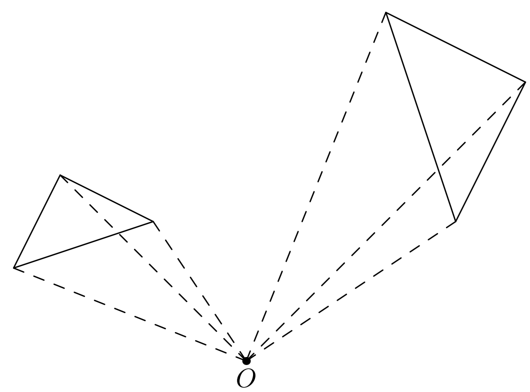


Figure 4: An example of a spiral similarity.

For instance, in the complex plane, if $O = 0$, then spiral similarities are described by multiplication by a nonzero complex number. That is, spiral similarities have the form $z \mapsto \alpha z$, where $\alpha \in \mathbb{C} \setminus \{0\}$. Here $|\alpha|$ is the dilation factor, and $\arg \alpha$ is the angle of rotation. It is easy to deduce from here that if the center of the spiral similarity is some other point, say z_0, then the transformation is given by $z \mapsto z_0 + \alpha(z - z_0)$ (why?).

Fact 4. Let A, B, C, D be four distinct point in the plane, such that $ABCD$ is not a parallelogram. Then there exists a unique spiral similarity that sends A to B, and C to D.

Proof. Let a, b, c, d be the corresponding complex numbers for the points A, B, C, D. We know that a spiral similarity has the form $\mathbf{T}(z) = z_0 + \alpha(z - z_0)$, where z_0 is the center of the spiral similarity, and α is data on the rotation and dilation. So we would like to find α and z_0 such that $\mathbf{T}(a) = c$ and $\mathbf{T}(b) = d$. This amount to solving the system

$$z_0 + \alpha(a - z_0) = c, \qquad z_0 + \alpha(b - z_0) = d.$$

Solving it, we see that the unique solution is

$$\alpha = \frac{c - d}{a - b}, \qquad z_0 = \frac{ad - bc}{a - b - c + d}.$$

Since $ABCD$ is not a parallelogram, we see that $a - b - c + d \neq 0$, so that this is the unique solution to the system. Hence there exists a unique spiral similarity that carries A to B and C to D. $\square$

[2] If you want to impress your friends with your mathematical vocabulary, a spiral similarity is sometimes called a *similitude*, and a dilation is sometimes called a *homothety*. (Actually, they are not quite exactly the same thing, but shhh!)

Exercise 4. How can you quickly determine the value of α in the above proof without even needing to set up the system of equations?

Exercise 5. Give a geometric argument why the spiral similarity, if it exists, must be unique. (Hint: suppose that $\mathbf{T}_1$ and $\mathbf{T}_2$ are two such spiral similarities, then what can you say about $\mathbf{T}_1 \circ \mathbf{T}_2^{-1}$?)

Now we come to the key result of this section. It gives a very simple and useful description of the center of a spiral similarity. It can be very useful in locating very subtle spiral similarities hidden in a geometry problem. Remember this fact!

(Very Useful) Fact 5. Let A, B, C, D be four distinct point in the plane, such that AC is not parallel to BD. Let lines AC and BD meet at X. Let the circumcircles of ABX and CDX meet again at O. Then O is the center of the unique spiral similarity that carries A to C and B to D.

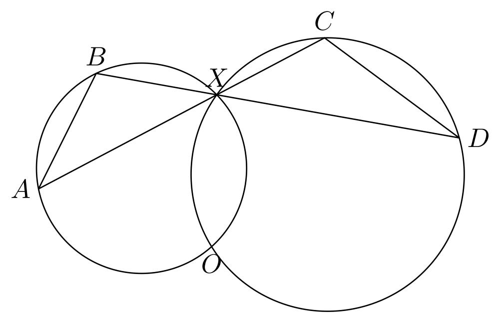


Figure 5: Diagram for Fact 5.

Proof. We give the proof only for the configuration shown above. Since $ABXO$ and $CDOX$ are cyclic, we have $\angle OBD = \angle OAC$ and $\angle OCA = \angle ODB$. It follows that triangles AOC and BOD are similar. Therefore, the spiral similarity centered at O that carries A to C must also carry B to D. $\square$

Exercise 6. Rewrite the above proof using directed angles mod π so that it works for all configurations.

Finally, it is is worth mentioning that spiral similarities often comes in pairs. If we can send AB to CD, then we can just as easily send AC to BD.

Fact 6. If O is the center of the spiral similarity that sends A to C and B to D, then O is also the center of the spiral similarity that sends A to B and C to D.

Proof. Since spiral similarity preserves angles at O, we have $\angle AOB = \angle COD$. Also, the dilation ratio of the first spiral similarity is $OC/OA = OD/OB$. So the rotation about with angle $\angle AOB = \angle COD$ and dilation with ratio $OB/OA = OD/OC$ sends A to B, and C to D, as desired. $\square$

Exercise 7. Deduce Fact 6 from Facts 2 and 5.

Now, let us apply these results to our configuration in Section 2.

Fact 7. Let M be the Miquel point of quadrilateral $ABCD$. Then M is the center of spiral similarity that sends AB to DC, as well as the center of the spiral similarity that sends AD to BC.

Exercise 8. Prove Fact 7.

Let us specialize to a cyclic quadrilateral, and continue the configuration in Fact 3

Fact 8. Let $ABCD$ be a cyclic quadrilateral with circumcenter O. Let lines AB and CD meet at Q, and lines DA and CB meet at R. Let M be the Miquel point of $ABCD$ (which lies on line QR, due to Fact 3). Then OM is perpendicular to QR.

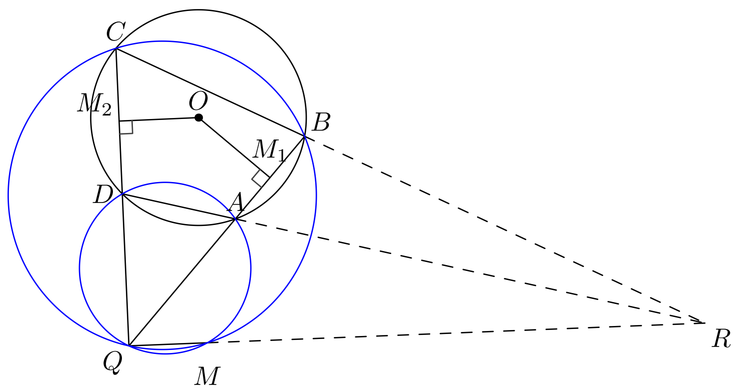


Figure 6: Diagram for the proof of Fact 8.

Proof. Let $\mathbf{T}$ denote the spiral similarity centered at M which sends A to D and B to C (Fact 7). Let M_1 and M_2 be the midpoint of AB and DC, respectively. Then $\mathbf{T}$ must send M_1 to M_2. So M is the center of unique spiral similarity that sends A to M_1 and D to M_2 (Fact 6), and thus it follows that M, M_1, M_2, Q are concyclic (Fact 5).

Since M_1 and M_2 are the midpoints of the chords AB and CD, we have $\angle OM_2Q = \angle OM_1Q$, and so O, M_1, M_2, Q are concyclic, and OQ is the diameter of the common circle. It follows that O, M, M_1, M_2, Q all lie on the circle with diameter OQ. In particular, $\angle OMQ = 90^\circ$, as desired. □

5 A Criterion for Orthogonality

In this section, we give another proof of Fact 8 and introduce a very useful computational criterion for orthogonality.

(Very Useful) Fact 9. Let A, B, C, D be points in the plane. Assume that $A \neq B$ and $C \neq D$. Then lines AB and CD are perpendicular if and only if $AC^2 + BD^2 = AD^2 + BC^2$.

Proof. The result follows immediately from the following identity.

$$(\vec{A}-\vec{C})\cdot(\vec{A}-\vec{C})+(\vec{B}-\vec{D})\cdot(\vec{B}-\vec{D})-(\vec{A}-\vec{D})\cdot(\vec{A}-\vec{D})-(\vec{B}-\vec{C})\cdot(\vec{B}-\vec{C}) = 2(\vec{B}-\vec{A})\cdot(\vec{C}-\vec{D}).$$

Note that the LHS is zero iff $AC^2 + BD^2 = AD^2 + BC^2$ and the RHS is zero iff $AB \perp CD$. □

Another proof of Fact 8. Let r be the circumradius of $ABCD$. Using Power of a Point on the circumcircles of $ABCD$ and $ABRM$, we get

$$QO^2 - r^2 = QA \cdot QB = QM \cdot QR = QM \cdot MR + QM^2$$

(the strategy here is to transfer all the data onto the line QR). Similarly, we have

$$RO^2 - r^2 = RA \cdot RD = RM \cdot RQ = QM \cdot MR + RM^2.$$

Subtracting the two relations, we get

$$QO^2 - RO^2 = QM^2 - RM^2,$$

and it thus follows from Fact 9 that OM is perpendicular to QR. □

6 Radical Axis

Given two circles in the plane, their *radical axis* is the locus of points of equal power to the two circles. It turns out that this is always a line. If the two circles intersect, then the radical axis is the line passing through the two intersection points (i.e., the common chord). If the two circles are tangent, then the radical axis is the common internal tangent.

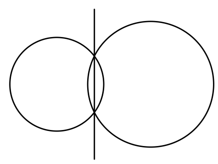


Figure 7: An example of a radical axis.

Exercise 9. Use Fact 9 to deduce that the radical axis is always a line.

It is well known (and easy to prove) that, given three distinct circles, their pairwise radical axes are either concurrent or all parallel. If the three radical axes meet at a common point, we say that the common intersection point is the *radical center* of the three circles.

For instance, using the setup from Fact 8, we see that BC is the radical axis of circles $ABCD$ and $BCQM$, AD is the radical axis of circles $ABCD$ and $ADQM$, and QM is the radical axes of circles $ADQM$ and $BCQM$. So the lines AD, BC, QM meet at common point, R, the radical center of the three circles: $ABCD, ADQM, BCQM$.

Fact 10. Use the setup from Fact 8. Points A, C, M, O are concyclic, and points B, D, M, O are concyclic.

Exercise 10. Prove Fact 10. (This is pretty easy angle chasing.)

Fact 11. Use the setup from Fact 8. The lines AC, BD, OM are concurrent.

Proof. Consider the three circles: $ABCD, AOCM, BODM$. Lines AC, BD, OM are the three radical axes, and thus they must concur. □

Exercise 11. Show that MO bisects $\angle CMA$ as well as $\angle BMD$.

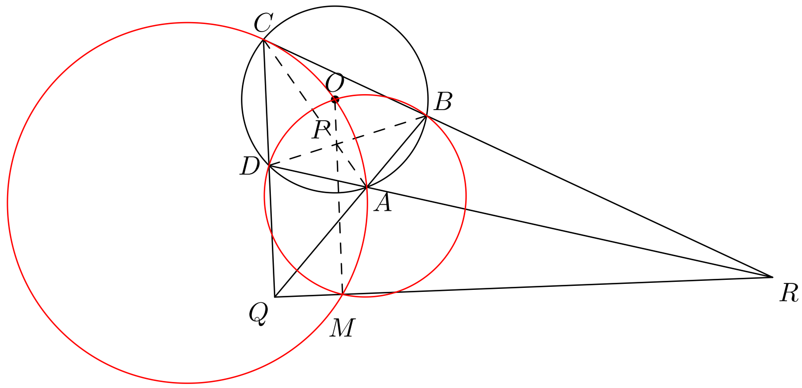


Figure 8: Diagram for the proof of Fact 11.

7 Inversion and Polarity

In this section, we assume some prior knowledge of inversion, as well as poles and polars. Here is a quick review:

Let $\mathcal{C}$ be a circle, with center O and radius r. The *inversion* with respect to $\mathcal{C}$ is a transformation (in fact, an involution) that sends a point P ($\neq O$) to a point P' on ray OP such that $OP \cdot OP' = r^2$. Inversions "switches lines and circles." Specifically, a line that pass through O gets sent to itself; a line not passing through O gets sent to a circle through O; a circle that pass through O gets sent to a line not passing through O; and a circle not passing through O gets sent to a (possible different) circle not passing through O.

Suppose that P ($\neq O$) is a point, and ℓ is a line passing through the inverse of P and also perpendicular to OP, then we say that ℓ is the *polar* of P, and that P is the *pole* of ℓ. Polar maps satisfy the principle of duality. For instance, the P lies on the polar of Q iff Q lies on the polar of P; ℓ_1 passes through the pole of ℓ_2 iff ℓ_2 passes through the pole of ℓ_2; three poles are collinear iff the three corresponding polars are concurrent.

Let us return to the configuration.

Fact 12. Let $ABCD$ be a cyclic quadrilateral with circumcenter O. Let AC and BD meet at P, lines AB and CD meet at Q, and lines DA and CB meet at R. Let M be the Miquel point of $ABCD$. Then P is the inverse of M with respect to the circumcircle of $ABCD$.

Proof. Since P is the intersection of AC and BD, under the inversion, it must be mapped to the intersection (other than O) of the circles OAC and OBD, which is M (Fact 10). □

Note that this gives another proof of Fact 11, which says that O, P, M are collinear.

Fact 13. The line QR is the polar of the point P.

Proof. This follows from Fact 8 and Fact 13. □

Given a circle, $\mathcal{C}$, we say that a triangle is *self-polar* if each side is the polar of the opposite vertex.

Now we are able to prove an extremely useful result in projective geometry.[3]

(Very Useful) Fact 14. The triangle PQR is self-polar with respect to the circumcircle of $ABCD$.

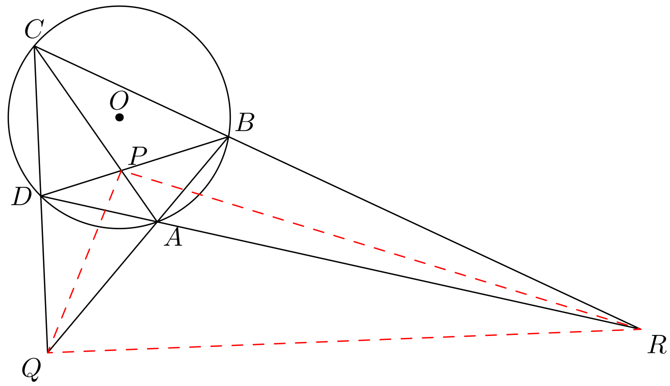


Figure 9: PQR is self-polar.

Proof. There is nothing in the proofs that required us to have A, B, C, D in that order on the circle. By permuting the relabels of A, B, C, D, we can deduce from Fact 13 that PR is the polar of Q, and PQ is the polar of R. This gives the desired result. □

Fact 15. O is the orthocenter of PQR.

Proof. This follows immediately from Fact 14, since $OX \perp \ell$ for any pole-polar pair (X, ℓ). □

8 Summary

This concludes our analysis of the diagram in Section 2. Here is a summary of the key results that came out of it. (Refer to Figure 1.)

Theorem. Let $ABCD$ be a cyclic quadrilateral with circumcenter O. Let AC and BD meet at P, lines AB and CD meet at Q, and lines DA and CB meet at R. Let line OP meet QR at M. Then

(a) The circumcircles of the following triangles all pass through M: QAD, QBC, RAB, RDC, AOC, BOD. (In particular, M is the Miquel point of the quadrilateral $ABCD$.)

(b) M is the center of the spiral similarity that carries A to B and D to C, and also the center of the spiral similarity that carries A to D and B to C.

(c) $OM \perp QR$. In fact, M is the inverse of P with respect to the circumcircle of $ABCD$.

(d) The triangle PQR is self-polar with respect to the circumcircle of $ABCD$.

Remember this configuration! Many olympiad geometry problems are basically just a portion of this one big diagram.

[3]For what it's worth, here's a very quick sketch of a proof of Fact 14 using projective geometry: Let line RP intersect AB and BC at E and F, respectively. By applying perspectivities from P and R, we find that $(A, B; E, Q) = (C, D; F, Q) = (B, A; F, Q)$, from which it follows that $(A, B; E, Q)$ and $(C, D; F, Q)$ are both harmonic. It follows that EF is the polar of Q, and hence PR is the polar of Q. Similarly we can show that QR is the polar of P, and PQ is the polar of Q. So PQR is self-polar.

9 Problems

0. Work through all the exercises.

1. (IMO 1985) A circle with center O passes through the vertices A and C of triangle ABC and intersects segments AB and BC again at distinct points K and N, respectively. The circumcircles of triangles ABC and KBN intersects at exactly two distinct points B and M. Prove that $\angle OMB = 90°$.

2. (China 1992) Convex quadrilateral $ABCD$ is inscribed in circle ω with center O. Diagonals AC and BD meet at P. The circumcircles of triangles ABP and CDP meet at P and Q. Assume that points O, P, and Q are distinct. Prove that $\angle OQP = 90°$.

3. (Russia 1999) A circle through vertices A and B of a triangle ABC meets side BC again at D. A circle through B and C meets side AB at E and the first circle again at F. Prove that if points A, E, D, C lie on a circle with center O, then $\angle BFO = 90°$.

4. Circles ω_1 and ω_2 meet at points O and M. Circle ω, centered at O, meet circles ω_1 and ω_2 in four distinct points A, B, C and D, such that $ABCD$ is a convex quadrilateral. Lines AB and CD meet at N_1. Lines AD and BC meet at N_2. Prove that $N_1N_2 \perp MO$.

5. (Russia 1995; Romanian TST 1996; Iran 1997) Consider a circle with diameter AB and center O, and let C and D be two points on this circle. The line CD meets the line AB at a point M satisfying $MB < MA$ and $MD < MC$. Let K be the point of intersection (different from O) of the circumcircles of triangles AOC and DOB. Show that $\angle MKO = 90°$.

6. (a) Let A, B, C, D be four points in the plane. Let lines AC and BD meet at P, lines AB and CD meet at Q, and lines BC and DA meet at R. Let the line through P parallel to QR meet lines AB and CD at X and Z. Show that P is the midpoint of XZ.

   (b) Use part (a) and Fact 8 to prove the Butterfly Theorem: Let $\mathcal{C}$ be a circle and let EF be a chord. Let P be the midpoint of EF, and let AC, BD be two other chords passing through P. Suppose that AB and CD meet EF at X and Z, respectively, then $PX = PZ$.

7. Let $ABCD$ be a cyclic quadrilateral with circumcenter O. Let lines AB and CD meet at R. Let ℓ denote the line through R perpendicular to OR. Prove that lines BD and AC meet on ℓ at points equidistant from R.

8. (USA TST 2007) Triangle ABC is inscribed in circle ω. The tangent lines to ω at B and C meet at T. Point S lies on ray BC such that $AS \perp AT$. Points B_1 and C_1 lies on ray ST (with C_1 in between B_1 and S) such that $B_1T = BT = C_1T$. Prove that triangles ABC and AB_1C_1 are similar to each other.

9. Let ABC be a triangle with incenter I. Points M and N are the midpoints of side AB and AC, respectively. Points D and E lie on lines AB and AC, respectively, such that $BD = CE = BC$. Line ℓ_1 pass through D and is perpendicular to line IM. Line ℓ_2 passes through E and is perpendicular to line IN. Let P be the intersection of lines ℓ_1 and ℓ_2. Prove that $AP \perp BC$.

10. (IMO 2005) Let $ABCD$ be a given convex quadrilateral with sides BC and AD equal in length and not parallel. Let E and F be interior points of the sides BC and AD respectively

such that $BE = DF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R. Consider all the triangles PQR as E and F vary. Show that the circumcircles of these triangles have a common point other than P.

11. A circle is inscribed in quadrilateral $ABCD$ so that it touches sides AB, BC, CD, DA at E, F, G, H respectively.
    (a) Show that lines AC, EF, GH are concurrent. In fact, they concur at the pole of BD.
    (b) Show that lines AC, BD, EG, FH are concurrent.

12. (China 1997) Let quadrilateral $ABCD$ be inscribed in a circle. Suppose lines AB and DC intersect at P and lines AD and BC intersect at Q. From Q, construct the two tangents QE and QF to the circle where E and F are the points of tangency. Prove that the three points P, E, F are collinear.

13. Let $ABCD$ be a cyclic quadrilateral with circumcenter O. Let lines AB and CD meet at E, AD and BC meet at F, and AC and BD meet at P. Furthermore, let EP and AD meet at K, and let M be the projection of O onto AD be M. Prove that $BCMK$ is cyclic.

14. (IMO Shortlist 2006) Points A_1, B_1 and C_1 are chosen on sides BC, CA, and AB of a triangle ABC, respectively. The circumcircles of triangles AB_1C_1, BC_1A_1, and CA_1B_1 intersect the circumcircle of triangle ABC again at points A_2, B_2, and C_2, respectively ($A_2 \neq A, B_2 \neq B$, and $C_2 \neq C$). Points A_3, B_3, and C_3 are symmetric to A_1, B_1, C_1 with respect to the midpoints of sides BC, CA, and AB, respectively. Prove that triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.

15. Euler point of a cyclic quadrilateral
    (a) Let $ABCD$ be a cyclic quadrilateral. Let H_A, H_B, H_C, H_D be the orthocenters of BCD, ACD, ABD, ABC, respectively. Show that $H_AH_BH_CH_D$ is the image of $ABCD$ under a reflection about some point E (i.e. a 180° rotation about E).

        Point E is called the *Euler point* of $ABCD$. (Aside: why is it called the Euler point?[4])
    (b) Show that E lies on the nine-point-circles of triangles ABC, ABD, ACD, BCD.
    (c) Show that E lies on the Simson line of triangle ABC and point D.
    (d) Show that E is also the Euler point of $H_AH_BH_CH_D$.
    (e) Let M_{XY} denote the midpoint of XY. Show that the perpendiculars from M_{AB} to CD, from M_{BC} to DA, from M_{CD} to AB, and from M_{DA} to BC, concur at E.

[4]Hint: Recall that the Euler point of a triangle is another name for the center of the nine-point-circle.

10 Hints

0. Really, they are not hard.
1. Find the configuration in the big diagram. Fact 8 is the key.
2. This is the same as the previous problem! (Why?)
3. We've done this too many times already!
4. Use Facts 8 and 10.
5. See previous problem. (Do we need AB to be a diameter?)
6. (a) There is a one-line solution using projective geometry (try a perspectivity at Q). (b) Use $OP \perp QR$.
7. Butterfly, metamorphized.
8. To see how this fits into the big diagram, try using BCC_1B_1 as the starting cyclic quadrilateral.
9. Repeatedly apply Fact 9.
10. Do you see a spiral similarity? Where is its center?
11. Use the self-polar diagonal triangle of $EFGH$.
12. Use the self-polar diagonal triangle of $ABCD$.
13. Through power of a point, it suffices to show that $FB \cdot FC = FM \cdot FK$.
14. Use Fact 5, and see that $\triangle C_2BA \sim \triangle C_1A_1B_1 \sim \triangle CA_3B_3$, and similarly with the other three vertices. Deduce that $\angle B_2A_2C_2 = \angle B_3A_3C_3$.
15. Complex numbers may be helpful. For (c), recall the following fact: the Simson line of ABC and D bisects DH_D. For (e), observe that a dilation of ratio 2 centered at B sends M_{AB} to A and E to H_B.