1

Forbidding a Subgraph

Chapter Highlights

- Turán problem: determine the maximum number of edges in an n-vertex H-free graph
- Mantel's and Turán's theorems: K_r-free
- Kővári–Sós–Turán theorem: $K_{s,t}$-free
- Erdős–Stone–Simonovits theorem: H-free for general H
- Dependent random choice technique: H-free for a bounded degree bipartite H
- Lower bound constructions of H-free graphs for bipartite H
- Algebraic constructions: matching lower bounds for $K_{2,2}$, $K_{3,3}$, and $K_{s,t}$ for t much larger than s, and also for C_4, C_6, C_{10}
- Randomized algebraic constructions

We begin by completely answering the following question.

Question 1.0.1 (Triangle-free graph)
What is the maximum number of edges in a triangle-free n-vertex graph?

We will see the answer shortly. More generally, we can ask about what happens if we replace "triangle" by an arbitrary subgraph. This is a foundational problem in extremal graph theory.

Definition 1.0.2 (Extremal number / Turán number)
We write $\mathbf{ex}(n, H)$ for the maximum number of edges in an n-vertex H-free graph. Here an ***H-free graph*** is a graph that does not contain H as a subgraph.

In this book, by H-free we always mean forbidding H as a subgraph, rather than as an induced subgraph. (See Notation and Conventions at the beginning of the book for the distinction.)

Question 1.0.3 (Turán problem)
Determine $\mathrm{ex}(n, H)$. Given a graph H, how does $\mathrm{ex}(n, H)$ grow as $n \to \infty$?

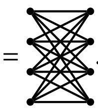
The Turán problem is one of the most basic problems in extremal graph theory. It is named after Pál Turán for his fundamental work on the subject. Research on this problem has led to many important techniques. We will see a fairly satisfactory answer to the Turán problem for nonbipartite graphs H. We also know the answer for a small number of bipartite graphs H. However, for nearly all bipartite graphs H, much mystery remains.

In the first part of the chapter, we focus on techniques for upper bounding $\mathrm{ex}(n, H)$. In the last few sections, we turn our attention to lower bounding $\mathrm{ex}(n, H)$ when H is a bipartite graph.

1.1 Forbidding a Triangle: Mantel's Theorem

We begin by answering Question 1.0.1: what is the maximum number of edges in an n-vertex triangle-free graph? This question was answered in the early 1900's by Willem Mantel, whose theorem is considered the starting point of extremal graph theory.

Let us partition the n vertices into two equal halves (differing by one if n is odd), and then put in all edges across the two parts. This is the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$, and is triangle-free. For example,

$K_{4,4} =$  .

The graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ has $\lfloor n/2 \rfloor \lceil n/2 \rceil = \lfloor n^2/4 \rfloor$ edges (one can check this equality by separately considering even and odd n).

Mantel (1907) proved that $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ has the greatest number of edges among all triangle-free graphs.

Theorem 1.1.1 (Mantel's theorem)
Every n-vertex triangle-free graph has at most $\lfloor n^2/4 \rfloor$ edges.

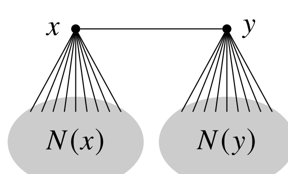
Using the notation of Definition 1.0.2, Mantel's theorem says that

$$\mathrm{ex}(n, K_3) = \left\lfloor \frac{n^2}{4} \right\rfloor .$$

Moreover, we will see that $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ is the unique maximizer of the number of edges among n-vertex triangle-free graphs.

We give two different proofs of Mantel's theorem, each illustrating a different technique.

First proof of Mantel's theorem. Let $G = (V, E)$ be a triangle-free graph with $|V| = n$ vertices and $|E| = m$ edges. For every edge xy of G, note that x and y have no common neighbors or else it would create a triangle.



Therefore, $\deg x + \deg y \leq n$, which implies that

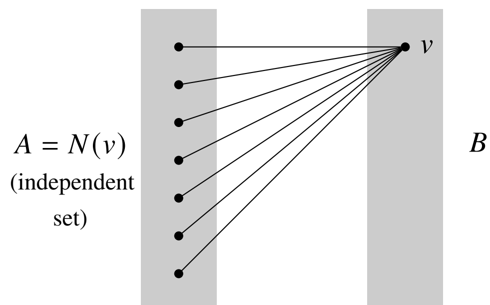
$$\sum_{xy \in E} (\deg x + \deg y) \leq mn.$$

On the other hand, note that for each vertex x, the term $\deg x$ appears once in the preceding sum for each edge incident to x, and so it appears a total of $\deg x$ times. We then apply the Cauchy–Schwarz inequality to get

$$\sum_{xy\in E}(\deg x + \deg y) = \sum_{x\in V}(\deg x)^2 \geq \frac{1}{n}\left(\sum_{x\in V}\deg x\right)^2 = \frac{(2m)^2}{n}.$$

Comparing the two inequalities, we obtain $(2m)^2/n \leq mn$, and hence $m \leq n^2/4$. Since m is an integer, we obtain $m \leq \lfloor n^2/4 \rfloor$, as claimed. □

Second proof of Mantel's theorem. Let $G = (V, E)$ be a triangle-free graph. Let v be a vertex of maximum degree in G. Since G is triangle-free, the neighborhood $N(v)$ of v is an independent set.



Partition $V = A \cup B$ where $A = N(v)$ and $B = V \setminus A$. Since v is a vertex of maximum degree, we have $\deg x \leq \deg v = |A|$ for all $x \in V$. Since A contains no edges, every edge of G has at least one endpoint in B. Therefore,

$$|E| \leq \sum_{x\in B}\deg x \leq |B| \max_{x\in B}\deg x \leq |A|\,|B| \leq \left(\frac{|A|+|B|}{2}\right)^2 = \frac{n^2}{4}, \tag{1.1}$$

as claimed. □

***Remark* 1.1.2** (The equality case in Mantel's theorem)**.** The second proof just presented shows that every n-vertex triangle-free graph with exactly $\lfloor n^2/4\rfloor$ edges must be isomorphic to $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$. Indeed, in (1.1), the inequality $|E| \leq \sum_{x\in B}\deg x$ is tight only if B is an independent set, the inequality $\sum_{x\in B}\deg x \leq |A|\,|B|$ is tight if B is complete to A, and $|A|\,|B| < \lfloor n^2/4\rfloor$ unless $|A| = |B|$ (if n is even) or $||A| - |B|| = 1$ (if n is odd).

(Exercise: also deduce the equality case from the first proof.)

In general, it is a good idea to keep the equality case in mind when following the proofs, or when coming up with your own proofs, to make sure you are not giving away too much at any step.

The next exercise can be solved by a neat application of Mantel's theorem.

Exercise 1.1.3. Let X and Y be independent and identically distributed random vectors in $\mathbb{R}^d$ according to some arbitrary probability distribution. Prove that

$$\mathbb{P}(|X+Y| \geq 1) \geq \frac{1}{2}\mathbb{P}(|X| \geq 1)^2.$$

Hint: Consider i.i.d. $X_1, \ldots, X_n$, and then let $n \to \infty$.

The next several exercises explore extensions of Mantel's theorem. It is useful to revisit the proof techniques.

Exercise 1.1.4 (Many triangles). Show that a graph with n vertices and m edges has at least

$$\frac{4m}{3n}\left(m - \frac{n^2}{4}\right) \text{ triangles.}$$

Exercise 1.1.5. Prove that every n-vertex nonbipartite triangle-free graph has at most $(n-1)^2/4 + 1$ edges.

Exercise 1.1.6 (Stability). Let G be an n-vertex triangle-free graph with at least $\lfloor n^2/4 \rfloor - k$ edges. Prove that G can be made bipartite by removing at most k edges.

Exercise 1.1.7. Show that every n-vertex triangle-free graph with minimum degree greater than $2n/5$ is bipartite.

Exercise 1.1.8*. Prove that every n-vertex graph with at least $\lfloor n^2/4 \rfloor + 1$ edges contains at least $\lfloor n/2 \rfloor$ triangles.

Exercise 1.1.9*. Let G be an n-vertex graph with $\lfloor n^2/4 \rfloor - k$ edges (here $k \in \mathbb{Z}$) and t triangles. Prove that G can be made bipartite by removing at most $k + 6t/n$ edges, and that this constant 6 is the best possible.

Exercise 1.1.10*. Prove that every n-vertex graph with at least $\lfloor n^2/4 \rfloor + 1$ edges contains some edge in at least $(1/6 - o(1))n$ triangles, and that this constant $1/6$ is the best possible.

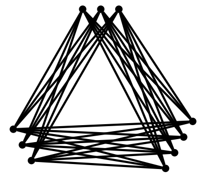
1.2 Forbidding a Clique: Turán's Theorem

We generalize Mantel's theorem from triangles to cliques.

Question 1.2.1 (K_{r+1}-free graph)
What is the maximum number of edges in a K_{r+1}-free graph on n vertices?

Construction 1.2.2 (Turán graph)
The ***Turán graph*** $\boldsymbol{T_{n,r}}$ is defined to be the complete n-vertex r-partite graph with part sizes differing by at most 1 (so each part has size $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$).

***Example* 1.2.3.** $T_{10,3} = K_{3,3,4}$:



Turán (1941) proved the following fundamental result.

Theorem 1.2.4 (Turán's theorem)
The Turán graph $T_{n,r}$ maximizes the number of edges among all n-vertex K_{r+1}-free graphs. It is also the unique maximizer.

The first part of the theorem says that

$$\mathrm{ex}(n, K_{r+1}) = e(T_{n,r}).$$

It is not too hard to give a precise formula for $e(T_{n,r})$, though there is a small, annoying dependence on the residue class of n mod r. The following bound is good enough for most purposes.

Exercise 1.2.5. Show that

$$e(T_{n,r}) \le \left(1 - \frac{1}{r}\right) \frac{n^2}{2},$$

with equality if and only if n is divisible by r.

Corollary 1.2.6 (Turán's theorem)

$$\mathrm{ex}(n, K_{r+1}) \le \left(1 - \frac{1}{r}\right) \frac{n^2}{2}.$$

Even when n is not divisible by r, the difference between $e(T_{n,r})$ and $(1 - 1/r)n^2/2$ is $O(nr)$. As we are generally interested in the regime when r is fixed, this difference is a negligible lower-order contribution. That is,

$$\mathrm{ex}(n, K_{r+1}) = \left(1 - \frac{1}{r} - o(1)\right) \frac{n^2}{2}, \quad \text{for fixed } r \text{ as } n \to \infty.$$

Every r-partite graph is automatically K_{r+1}-free. Let us first consider an easy special case of the problem.

Lemma 1.2.7 (Maximum number of edges in an r-partite graph)
Among n-vertex r-partite graphs, $T_{n,r}$ is the unique graph with the maximum number of edges.

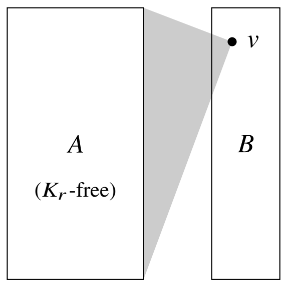
Proof. Suppose we have an n-vertex r-partite graph with the maximum possible number of edges. It should be a complete r-partite graph. If there were two vertex parts A and B with $|A| + 2 \le |B|$, then moving a vertex from B (the larger part) to A (the smaller part) would increase the number of edges by $(|A| + 1)(|B| - 1) - |A|\,|B| = |B| - |A| - 1 > 0$. Thus all the vertex parts must have sizes within one of each other. The Turán graph $T_{n,r}$ is the unique such graph. □

We will see three proofs of Turán's theorem. The first proof extends our second proof of Mantel's theorem.

First proof of Turán's theorem. We prove by induction on r. The case $r = 1$ is trivial, as a K_2-free graph is empty. Now assume $r > 1$ and that $\mathrm{ex}(n, K_r) = e(T_{n,r-1})$ for every n.

Let $G = (V, E)$ be a K_{r+1}-free graph. Let v be a vertex of maximum degree in G. Since G is K_{r+1}-free, the neighborhood $A = N(v)$ of v is K_r-free. So, by the induction hypothesis,

$$e(A) \le \mathrm{ex}(|A|, K_r) = e(T_{|A|,r-1}).$$



Let $B = V \setminus A$. Since v is a vertex of maximum degree, we have $\deg x \le \deg v = |A|$ for all $x \in V$. So, the number of edges with at least one vertex in B is

$$e(A, B) + e(B) \le \sum_{x \in B} \deg x \le |B| \max_{x \in B} \deg x \le |A|\,|B|.$$

Thus

$$e(G) = e(A) + e(A, B) + e(B) \le e(T_{|A|,r-1}) + |A|\,|B| \le e(T_{n,r}),$$

where the final step follows from the observation that $e(T_{|A|,r-1}) + |A|\,|B|$ is the number of edges in an n-vertex r-partite graph (with part of size $|B|$ and the remaining vertices equitably partitioned into $r - 1$ parts) and Lemma 1.2.7.

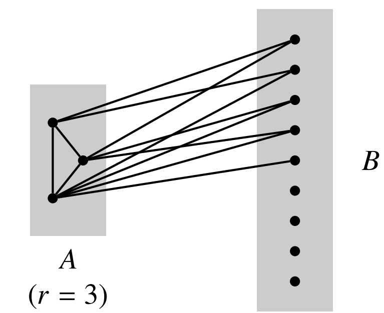
To have equality in each of the preceding steps, B must be an independent set (or else $\sum_{y \in B} \deg(y) < |A|\,|B|$) and A must induce $T_{|A|,r-1}$, so that G is r-partite. We knew from Lemma 1.2.7 that the Turán graph $T_{n,r}$ uniquely maximizes the number of edges among r-partite graphs. □

The second proof starts out similarly to our first proof of Mantel's theorem. Recall that in Mantel's theorem, the initial observation was that in a triangle-free graph, given an edge, its two endpoints must have no common neighbors (or else they form a triangle). Generalizing, in a K_4-free graph, given a triangle, its three vertices have no common neighbor. The rest of the proof proceeds somewhat differently from earlier. Instead of summing over all edges as we did before, we remove the triangle and apply induction to the rest of the graph.

Second proof of Turán's theorem. We fix r and proceed by induction on n. The statement is trivial for $n \le r$, as the Turán graph is the complete graph $K_n = T_{n,r}$ and thus maximizes the number of edges.

Now, assume that $n > r$ and that Turán's theorem holds for all graphs on fewer than n

vertices. Let $G = (V, E)$ be an n-vertex K_{r+1}-free graph with the maximum possible number of edges. By the maximality assumption, G contains K_r as a subgraph, since otherwise we could add an edge to G and it would still be K_{r+1}-free. Let A be the vertex set of an r-clique in G, and let $B := V \setminus A$.



Since G is K_{r+1}-free, every $x \in B$ has at most $r - 1$ neighbors in A. So

$$e(A, B) = \sum_{y \in B} \deg(y, A) \le \sum_{y \in B} (r-1) = (r-1)(n-r).$$

We have

$$\begin{aligned} e(G) &= e(A) + e(A, B) + e(B) \\ &\le \binom{r}{2} + (r-1)(n-r) + e(T_{n-r,r}) = e(T_{n,r}), \end{aligned}$$

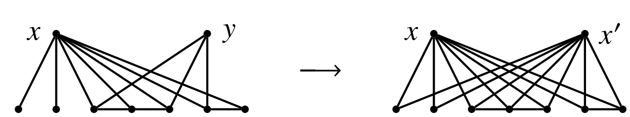
where the inequality uses the induction hypothesis on $G[B]$, which is K_{r+1}-free, and the final equality can be seen by removing a K_r from $T_{n,r}$.

Finally, let us check when equality occurs. To have equality in each of the preceding steps, the subgraph induced on B must be $T_{n-r,r}$ by induction. To have $e(A) = \binom{r}{2}$, A must induce a clique. To have $e(A, B) = (r-1)(n-r)$, every vertex of B must be adjacent to all but one vertex in A. Also, two vertices x, y lying in distinct parts of $G[B] \cong T_{n-r,r}$ cannot "miss" the same vertex v of A, or else $A \cup \{x, y\} \setminus \{v\}$ would be an K_{r+1}-clique. This then forces G to be $T_{n,r}$. □

The third proof uses a method known as **Zykov symmetrization**. The idea here is that if a K_{r+1}-free graph that is a not a Turán graph, then we should be able make some local modifications (namely replacing a vertex by a clone of another vertex) to get another K_{r+1}-free with strictly more edges.

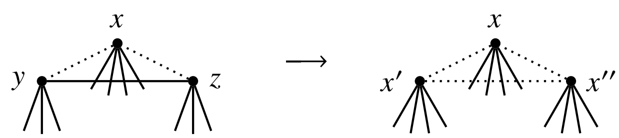
Third proof of Turán's theorem. As before, let G be an n-vertex K_{r+1}-free graph with the maximum possible number of edges.

We claim that if x and y are nonadjacent vertices, then $\deg x = \deg y$. Indeed, suppose $\deg x > \deg y$. We can modify G by removing y and adding in a clone of x (a new vertex x' with the same neighborhood as x but not adjacent to x), as illustrated below.



The resulting graph would still be K_{r+1}-free (since a clique cannot contain both x and its clone) and has strictly more edges than G, thereby contradicting the assumption that G has the maximum possible number of edges.

Suppose x is nonadjacent to both y and z in G. We claim that y and z must be nonadjacent. We just saw that $\deg x = \deg y = \deg z$. If yz is an edge, then by deleting y and z from G and adding two clones of x, we obtain a K_{r+1}-free graph with one more edge than G. This would contradict the maximality of G.



Therefore, nonadjacency is an equivalence relation among vertices of G. So the complement of G is a union of cliques. Hence G is a complete multipartite graph, which has at most r parts since G is K_{r+1}-free. Among all complete r-partite graphs, the Turán graph $T_{n,r}$ is the unique graph that maximizes the number of edges, by Lemma 1.2.7. Therefore, G is isomorphic to $T_{n,r}$. □

The last proof we give in this section uses the probabilistic method. This probabilistic proof was given in the book *The Probabilistic Method* by Alon and Spencer, though the key inequality is due earlier to Caro and Wei. Below, we prove Turán's theorem in the formulation of Corollary 1.2.6, i.e, $\mathrm{ex}(n, K_{r+1}) \le (1 - 1/r)n^2/2$. A more careful analysis of the proof can yield the stronger statement of Theorem 1.2.4, which we omit.

Fourth proof of Turán's theorem (Corollary 1.2.6). Let $G = (V, E)$ be an n-vertex, K_{r+1}-free graph. Consider a uniform random ordering of the vertices. Let

$$X = \{v \in V : v \text{ is adjacent to all earlier vertices in the random ordering}\}.$$

Then X is a clique. Since the ordering was chosen uniformly at random,

$$\mathbb{P}(v \in X) = \mathbb{P}(v \text{ appears before all its nonneighbors}) = \frac{1}{n - \deg v}.$$

Since G is K_{r+1}-free, $|X| \le r$. So by linearity of expectations

$$\begin{aligned} r \ge \mathbb{E}\,|X| &= \sum_{v \in V} \mathbb{P}(v \in X) \\ &= \sum_{v \in V} \frac{1}{n - \deg v} \ge \frac{n}{n - (\sum_{v \in V} \deg v)/n} = \frac{n}{n - 2m/n}. \end{aligned}$$

Rearranging gives

$$m \le \left(1 - \frac{1}{r}\right)\frac{n^2}{2}. \qquad \square$$

In Chapter 5, we will see another proof of Turán's theorem using a method known as **graph Lagrangians**.

Exercise 1.2.8. Let G be a K_{r+1}-free graph. Prove that there exists an r-partite graph H on the same vertex set as G such that $\deg_H(x) \ge \deg_G(x)$ for every vertex x (here $\deg_H(x)$ is the degree of x in H, and likewise with $\deg_G(x)$ for G). Give another proof of Turán's theorem from this fact.

The following exercise is an extension of Exercise 1.1.6.

Exercise 1.2.9* (Stability). Let G be an n-vertex K_{r+1}-free graph with at least $e(T_{n,r}) - k$ edges, where $T_{n,r}$ is the Turán graph. Prove that G can be made r-partite by removing at most k edges.

The next exercise is a neat geometric application of Turán's theorem.

Exercise 1.2.10. Let S be a set of n points in the plane, with the property that no two points are at distance greater than 1. Show that S has at most $\lfloor n^2/3 \rfloor$ pairs of points at distance greater than $1/\sqrt{2}$. Also, show that the bound $\lfloor n^2/3 \rfloor$ is tight (i.e., cannot be improved).

1.3 Turán Density and Supersaturation

Turán's theorem exactly determines $\mathrm{ex}(n, H)$ when H is a clique. Such precise answers are actually quite rare in extremal graph theory. We are often content with looser bounds and asymptotics.

We will go on to bound $\mathrm{ex}(n, H)$ for other values of H. But for now, let us take a short detour and think about the structure of the problem.

Turán Density

In this chapter, we will define the ***edge density*** of a graph G to be

$$e(G) \Big/ \binom{v(G)}{2}.$$

So the edge density of a clique is 1. Later in the book, we will consider a different normalization $2e(G)/v(G)^2$ for edge density, which is more convenient for other purposes. When $v(G)$ is large, there is no significant difference between the two choices.

Next, we use an averaging/sampling argument to show that $\mathrm{ex}(n, H)/\binom{n}{2}$ is nonincreasing in n.

Proposition 1.3.1 (Monotonicity of Turán numbers)
For every graph H and positive integer n,

$$\frac{\mathrm{ex}(n+1, H)}{\binom{n+1}{2}} \le \frac{\mathrm{ex}(n, H)}{\binom{n}{2}}.$$

Proof. Let G be an H-free graph on $n+1$ vertices. For each n-vertex subset S of $V(G)$, since $G[S]$ is also H-free, we have

$$\frac{e(G[S])}{\binom{n}{2}} \le \frac{\mathrm{ex}(n, H)}{\binom{n}{2}}.$$

Varying S uniformly over all n-vertex subsets of $V(G)$, the left-hand side averages to the edge density of G by linearity of expectations. (Check this.) It follows that

$$\frac{e(G)}{\binom{n+1}{2}} \le \frac{\mathrm{ex}(n,H)}{\binom{n}{2}}.$$

The claim then follows. □

For every fixed H, the sequence $\mathrm{ex}(n,H)/\binom{n}{2}$ is nonincreasing and bounded between 0 and 1. It follows that it approaches a limit.

Definition 1.3.2 (Turán density)
The ***Turán density*** of a graph H is defined to be

$$\boldsymbol{\pi(H)} := \lim_{n\to\infty} \frac{\mathrm{ex}(n,H)}{\binom{n}{2}}.$$

Here are some additional equivalent definitions of Turán density:

- $\pi(H)$ is smallest real number so that for every $\varepsilon > 0$ there is some $n_0 = n_0(H,\varepsilon)$ so that for every $n \ge n_0$, every n-vertex graph with at least $(\pi(H)+\varepsilon)\binom{n}{2}$ edges contains H as a subgraph;
- $\pi(H)$ is the smallest real number so that every n-vertex H-free graph has edge density $\le \pi(H) + o(1)$.

Recall, from Turán's theorem, that

$$\mathrm{ex}(n, K_{r+1}) = \left(1 - \frac{1}{r} - o(1)\right)\frac{n^2}{2}, \quad \text{for fixed } r \text{ as } n \to \infty,$$

which is equivalent to

$$\pi(K_{r+1}) = 1 - \frac{1}{r}.$$

In the next couple of sections we will prove the **Erdős–Stone–Simonovits theorem**, which determines the Turán density for every graph H:

$$\pi(H) = 1 - \frac{1}{\chi(H) - 1},$$

where $\chi(H)$ is the chromatic number of H. It should be surprising that the Turán density of H depends only on the chromatic number of H.

With the Erdős–Stone–Simonovits theorem, it may seem as if the Turán problem is essentially understood, but actually this would be very far from the truth. We will see in the next section that $\pi(H) = 0$ for every bipartite graph H. In other words $\mathrm{ex}(n,H) = o(n^2)$. Actual asymptotics growth rate of $\mathrm{ex}(n,H)$ is often unknown.

In a different direction, the generalization to hypergraphs, while looking deceptively similar, turns out to be much more difficult, and very little is known here.

***Remark* 1.3.3** (Hypergraph Turán problem)**.** Generalizing from graphs to hypergraphs, given an r-uniform hypergraph H, we write $\mathrm{ex}(n,H)$ for the maximum number of edges in an n-vertex r-uniform hypergraph that does not contain H as a subgraph. A straightforward extension of Proposition 1.3.1 gives that $\mathrm{ex}(n,H)/\binom{n}{r}$ is a nonincreasing function of n, for each fixed H. So we can similarly define the hypergraph Turán density

$$\pi(H) := \lim_{n\to\infty} \frac{\mathrm{ex}(n,H)}{\binom{n}{r}}.$$

The exact value of $\pi(H)$ is known in very few cases. It is a major open problem to determine $\pi(H)$ when H is the complete 3-uniform hypergraph on four vertices (also known as a tetrahedron), and more generally when H is a complete hypergraph.

Supersaturation

We know from Mantel's theorem that any n-vertex graph G with $> n^2/4$ edges must contain a triangle. What if G has a lot more edges? It turns out that G must have a lot of triangles. In particular, an n-vertex graph with $> (1/4+\varepsilon)n^2$ edges must have at least δn^3 triangles for some constant $\delta > 0$ depending on $\varepsilon > 0$. This is indeed a lot of triangles, since there are could only be at most $O(n^3)$ triangles no matter what. (Exercise 1.1.4 asks you to give a more precise quantitative lower bound on the number of triangles. The optimal dependence of δ on ε is a difficult problem that we will discuss in Chapter 5.)

It turns out there is a general phenomenon in combinatorics where once some density crosses an existence threshold (e.g., the Turán density is the threshold for H-freeness), it will be possible to find not just one copy of the desired object, but in fact lots and lots of copies. This fundamental principle, called **supersaturation**, is useful for many applications, including in our upcoming determination of $\pi(H)$ for general H.

Theorem 1.3.4 (Supersaturation)
For every $\varepsilon > 0$ and graph H there exist some $\delta > 0$ and n_0 such that every graph on $n \ge n_0$ vertices with at least $(\pi(H)+\varepsilon)\binom{n}{2}$ edges contains at least $\delta n^{v(H)}$ copies of H as a subgraph.

Equivalently: every n-vertex graph with $o(n^{v(H)})$ copies of H has edge density $\le \pi(H) + o(1)$ (here H is fixed). The **sampling argument** in the following proof is useful in many applications.

Proof. By the definition of the Turán density, there exists some n_0 (depending on H and ε) such that every n_0-vertex graph with at least $(\pi(H)+\varepsilon/2)\binom{n_0}{2}$ edges contains H as a subgraph.

Let $n \ge n_0$ and G be an n-vertex graph with at least $(\pi(H)+\varepsilon)\binom{n}{2}$ edges. Let S be an n_0-element subset of $V(G)$, chosen uniformly at random. Let X denote the edge density of $G[S]$. By averaging, $\mathbb{E}X$ equals the edge density of G, and so $\mathbb{E}X \ge \pi(H)+\varepsilon$. Then $X \ge \pi(H)+\varepsilon/2$ with probability $\ge \varepsilon/2$ (or else $\mathbb{E}X$ could not be as large as $\pi(H)+\varepsilon$). So, from the previous paragraph, we know that with probability $\ge \varepsilon/2$, $G[S]$ contains a copy of H. This gives us $\ge (\varepsilon/2)\binom{n}{n_0}$ copies of H, but each copy of H may be counted up to $\binom{n-v(H)}{n_0-v(H)}$ times. Thus, the number of copies of H in G is

$$\ge \frac{(\varepsilon/2)\binom{n}{n_0}}{\binom{n-v(H)}{n_0-v(H)}} = \Omega_{H,\varepsilon}(n^{v(H)}). \qquad \square$$

Exercise 1.3.5 (Supersaturation for hypergraphs)**.** Let H be an r-uniform hypergraph with hypergraph Turán density $\pi(H)$. Prove that every n-vertex r-uniform hypergraph with $o(n^{v(H)})$ copies of H has at most $(\pi(H) + o(1))\binom{n}{r}$ edges.

Exercise 1.3.6 (Density Ramsey)**.** Prove that for every s and r, there is some constant $c > 0$ so that for every sufficiently large n, if the edges of K_n are colored using r colors, then at least c fraction of all copies of K_s are monochromatic.

Exercise 1.3.7 (Density Szemerédi)**.** Let $k \geq 3$. Assuming Szemerédi's theorem for k-term arithmetic progressions (i.e., every subset of $[N]$ without a k-term arithmetic progression has size $o(N)$), prove the following density version of Szemerédi's theorem:

For every $\delta > 0$ there exist $c > 0$ and N_0 (both depending only on k and δ) such that for every $A \subseteq [N]$ with $|A| \geq \delta N$ and $N \geq N_0$, the number of k-term arithmetic progressions in A is at least cN^2.

1.4 Forbidding a Complete Bipartite Graph: Kővári–Sós–Turán Theorem

In this section, we provide an upper bound on $\mathrm{ex}(n, K_{s,t})$, the maximum number of edges in an n-vertex $K_{s,t}$-free graph. It is a major open problem to determine the asymptotic growth of $\mathrm{ex}(n, K_{s,t})$. For certain pairs (s, t) the answer is known, as we will discuss later in the chapter.

Problem 1.4.1 (Zarankiewicz problem)

Determine $\mathrm{ex}(n, K_{s,t})$, the maximum number of edges in an n-vertex $K_{s,t}$-free graph.

Zarankiewicz (1951) originally asked a related problem: determine the maximum number of 1s in an $m \times n$ matrix without an $s \times t$ submatrix with all entries 1.

The main theorem of this section is the fundamental result due to Kővári, Sós, and Turán (1954). We will refer to it as the **KST theorem**, which stands both for its discoverers, as well as for the forbidden subgraph $K_{s,t}$.

Theorem 1.4.2 (Kővári–Sós–Turán theorem – "KST theorem")

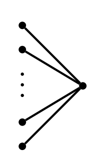
For positive integers $s \leq t$, there exists some constant $C = C(s, t)$, such that, for all n,

$$\mathrm{ex}(n, K_{s,t}) \leq Cn^{2-1/s}.$$

The proof proceeds by double counting.

Proof. Let G be an n-vertex $K_{s,t}$-free graph with m edges. Let

$$X = \text{number of copies of } K_{s,1} \text{ in } G.$$



(When $s = 1$, we set $X = 2e(G)$.) The strategy is to count X in two ways. First we count $K_{s,1}$ by first embedding the "left" s vertices of $K_{s,1}$. Then we count $K_{s,1}$ by first embedding the "right" single vertex of $K_{s,1}$.

Upper bound on X. Since G is $K_{s,t}$-free, every s-vertex subset of G has $\le t-1$ common neighbors. Therefore,

$$X \le \binom{n}{s}(t-1).$$

Lower bound on X. For each vertex v of G, there are exactly $\binom{\deg v}{s}$ ways to pick s of its neighbors to form a $K_{s,1}$ as a subgraph. Therefore

$$X = \sum_{v \in V(G)} \binom{\deg v}{s}.$$

To obtain a lower bound on this quantity in terms of the number of edges m of G, we use a standard trick of viewing $\binom{x}{s}$ as a convex function on the reals, namely, letting

$$f_s(x) = \begin{cases} x(x-1)\cdots(x-s+1)/s! & \text{if } x \ge s-1 \\ 0 & x < s-1. \end{cases}$$

Then $f_s(x) = \binom{x}{s}$ for all nonnegative integers x. Furthermore f_s is a convex function. Since the average degree of G is $2m/n$, it follows by convexity that

$$X = \sum_{v \in V(G)} f_s(\deg v) \ge n f_s\left(\frac{2m}{n}\right).$$

(It would be a sloppy mistake to lower bound X by $n\binom{2m/n}{s}$.)

Combining the upper bound and the lower bound. We find that

$$n f_s\left(\frac{2m}{n}\right) \le X \le \binom{n}{s}(t-1).$$

Since $f_s(x) = (1+o(1))x^s/s!$ for $x \to \infty$ and fixed s, we find that, as $n \to \infty$,

$$\frac{n}{s!}\left(\frac{2m}{n}\right)^s \le (1+o(1))\frac{n^s}{s!}(t-1).$$

Therefore,

$$m \le \left(\frac{(t-1)^{1/s}}{2} + o(1)\right) n^{2-1/s}. \qquad \square$$

The final bound in the proof gives us a somewhat more precise estimate than stated in Theorem 1.4.2. Let us record it here for future reference.

Theorem 1.4.3 (KST theorem)
Fix positive integers $s \le t$. Then, as $n \to \infty$,

$$\mathrm{ex}(n, K_{s,t}) \le \left(\frac{(t-1)^{1/s}}{2} + o(1)\right) n^{2-1/s}.$$

It has been long conjectured that the KST theorem is tight up to a constant factor.

Conjecture 1.4.4 (Tightness of KST bound)
For positive integers $s \le t$, there exists a constant $c = c(s,t) > 0$ such that for all $n \ge 2$,

$$\operatorname{ex}(n, K_{s,t}) \ge cn^{2-1/s}.$$

(In other words, $\operatorname{ex}(n, K_{s,t}) = \Theta_{s,t}(n^{2-1/s})$.)

In the final sections of this chapter, we will produce some constructions showing that Conjecture 1.4.4 is true for $K_{2,t}$ and $K_{3,t}$. We also know that the conjecture is true if t is much larger than s. The first open case of the conjecture is $K_{4,4}$.

Here is an easy consequence of the KST theorem.

Corollary 1.4.5
For every bipartite graph H, there exists some constant $c > 0$ so that $\operatorname{ex}(n, H) = O_H(n^{2-c})$.

Proof. Suppose the two vertex parts of H have sizes s and t, with $s \le t$. Then $H \subseteq K_{s,t}$. And thus every n-vertex H-free graph is also $K_{s,t}$-free, and thus has $O_{s,t}(n^{2-1/s})$ edges. □

In particular, the Turán density $\pi(H)$ of every bipartite graph H is zero.

The KST theorem gives a constant c in the preceding corollary that depends on the number of vertices on the smaller part of H. In Section 1.7, we will use the dependent random choice technique to give a proof of the corollary showing that c only has to depend on the maximum degree of H.

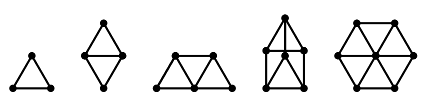
Geometric Applications of the KST Theorem

The following famous problem was posed by Erdős (1946).

Question 1.4.6 (Erdős unit distance problem)
What is the maximum number of unit distances formed by a set of n points in $\mathbb{R}^2$?

In other words, given n distinct points in the plane, at most how many pairs of these points can be exactly distance 1 apart? We can draw a graph with these n points as vertices, with edges joining points exactly a unit distance apart.

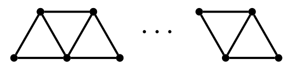
To get a feeling for the problem, let us play with some constructions. For small values of n, it is not hard to check by hand that the following configurations are optimal.



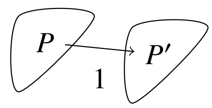
$n =$ 3 4 5 6 7

What about for larger values of n? If we line up the n points equally spaced on a line, we get $n - 1$ unit distances.

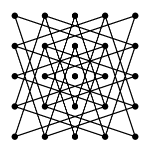
We can be a bit more efficient by chaining up triangles. The following construction gives us $2n - 3$ unit distances.



The construction for $n = 6$ looks like it was obtained by copying and translating a unit triangle. We can generalize this idea to obtain a recursive construction. Let $f(n)$ denote the maximum number of unit distances formed by n points in the plane. Given a configuration P with $\lfloor n/2 \rfloor$ points that has $f(\lfloor n/2 \rfloor)$ unit distances, we can copy P and translate it by a generic unit vector to get P'. The configuration $P \cup P'$ has at least $2f(\lfloor n/2 \rfloor) + \lfloor n/2 \rfloor$ unit distances. We can solve the recursion to get $f(n) \gtrsim n \log n$.



Now we take a different approach to obtain an even better construction. Take a square grid with $\lfloor \sqrt{n} \rfloor \times \lfloor \sqrt{n} \rfloor$ vertices. Instead of choosing the distance between adjacent points as the unit distance, we can scale the configuration so that $\sqrt{r}$ becomes the "unit" distance for some integer r. As an illustration, here is an example of a 5×5 grid with $r = 10$.



It turns out that by choosing the optimal r as a function of n, we can get at least

$$n^{1+c/\log\log n}$$

unit distances, where $c > 0$ is some absolute constant. The proof uses analytic number theory, which we omit as it would take us too far afield. The basic idea is to choose r to be a product of many distinct primes that are congruent to 1 modulo 4, so that r can be represented as a sum of two squares in many different ways, and then estimate the number of such ways.

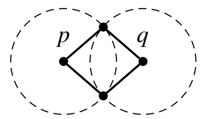
It is conjectured that the last construction just discussed is close to optimal.

Conjecture 1.4.7 (Erdős unit distance conjecture)
Every set of n points in $\mathbb{R}^2$ has at most $n^{1+o(1)}$ unit distances.

The KST theorem can be used to prove the following upper bound on the number of unit distances.

Theorem 1.4.8 (Upper bound on the unit distance problem)
Every set of n points in $\mathbb{R}^2$ has $O(n^{3/2})$ unit distances.

Proof. Every unit distance graph is $K_{2,3}$-free. Indeed, for every pair of distinct points, there are at most two other points that are at a unit distance from both points.



So, the number of edges is at most $\mathrm{ex}(n, K_{2,3}) = O(n^{3/2})$ by Theorem 1.4.2. □

There is a short proof of a better bound of $O(n^{4/3})$ using the crossing number inequality (see Section 8.2), and this is best known upper bound to date.

Erdős (1946) also asked the following related question.

Question 1.4.9 (Erdős distinct distances problem)
What is the minimum number of distinct distances formed by n points in $\mathbb{R}^2$?

Let $g(n)$ denote the answer. The asymptotically best construction for the minimum number of distinct distances is also a square grid, same as earlier. It can be shown that a square grid with $\lfloor \sqrt{n} \rfloor \times \lfloor \sqrt{n} \rfloor$ points has on the order of $n/\sqrt{\log n}$ distinct distances. This is conjectured to be optimal (i.e., $g(n) \gtrsim n/\sqrt{\log n}$).

Let $f(n)$ denote the maximum number of unit distances among n points in the answer. We have $f(n)g(n) \geq \binom{n}{2}$, since each distance occurs at most $f(n)$ times. So an upper bound on $f(n)$ gives a lower bound on $g(n)$ (but not conversely).

A breakthrough on the distinct distances problem was obtained by Guth and Katz (2015).

Theorem 1.4.10 (Guth–Katz distinct distances theorem)
A set of n points in $\mathbb{R}^2$ form $\Omega(n/\log n)$ distinct distances.

In other words, $g(n) \gtrsim n/\log n$, thereby matching the upper bound example up to a factor of $O(\sqrt{\log n})$. The Guth–Katz proof is quite sophisticated. It uses tools ranging from the polynomial method to algebraic geometry.

Exercise 1.4.11. Show that a C_4-free bipartite graph between two vertex parts of sizes a and b has at most $ab^{1/2} + b$ edges.

Exercise 1.4.12 (Density KST)**.** Prove that for every pair of positive integers $s \leq t$, there are constants $C, c > 0$ such that every n-vertex graph with $p\binom{n}{2}$ edges contains at least $cp^{st}n^{s+t}$ copies of $K_{s,t}$, provided that $p \geq Cn^{-1/s}$.

The next exercise asks you to think about the quantitative dependencies in the proof of the KST theorem.

Exercise 1.4.13. Show that for every $\varepsilon > 0$, there exists $\delta > 0$ such that every graph with n vertices and at least εn^2 edges contains a copy of $K_{s,t}$ where $s \geq \delta \log n$ and $t \geq n^{0.99}$.

The next exercise illustrates a bad definition of density of a subset of $\mathbb{Z}^2$ (it always ends up being either 0 or 1).

Exercise 1.4.14 (How not to define density)**.** Let $S \subseteq \mathbb{Z}^2$. Define

$$d_k(S) = \max_{\substack{A,B \subseteq \mathbb{Z} \\ |A|=|B|=k}} \frac{|S \cap (A \times B)|}{|A|\,|B|}.$$

Show that $\lim_{k\to\infty} d_k(S)$ exists and is always either 0 or 1.

1.5 Forbidding a General Subgraph: Erdős–Stone–Simonovits Theorem

Turán's theorem tells us that

$$\operatorname{ex}(n, K_{r+1}) = \left(1 - \frac{1}{r} - o(1)\right)\frac{n^2}{2} \quad \text{for fixed } r.$$

The KST theorem implies that

$$\operatorname{ex}(n, H) = o(n^2) \quad \text{for any fixed bipartite graph } H.$$

In this section, we extend these results and determine $\operatorname{ex}(n, H)$, up to an $o(n^2)$ error term, for every graph H. In other words, we will compute the Turán density $\pi(H)$.

Initially it seems possible that the Turán density $\pi(H)$ might depend on H in some complicated way. It turns out that it only depends on the ***chromatic number*** $\boldsymbol{\chi(H)}$ of H, which is the smallest number of colors needed to color the vertices of H such that no two adjacent vertices receive the same color.

Suppose $\chi(H) = r$. Then H cannot be a subgraph of any $(r-1)$-partite graph. In particular, the Turán graph $T_{n,r-1}$ is H-free. (Recall from Construction 1.2.2 that $T_{n,r-1}$ is the complete $(r-1)$-partite graph with n vertices divided into nearly equal parts.) Therefore,

$$\operatorname{ex}(n, H) \geq e(T_{n,r-1}) = \left(1 - \frac{1}{r-1} + o(1)\right)\frac{n^2}{2}.$$

The main theorem of this section, which follows, is a matching upper bound.

Theorem 1.5.1 (Erdős–Stone–Simonovits theorem)
For every graph H, as $n \to \infty$,

$$\operatorname{ex}(n, H) = \left(1 - \frac{1}{\chi(H) - 1} + o(1)\right)\frac{n^2}{2}.$$

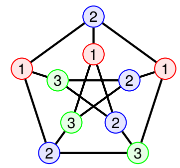
In other words, the Turán density of H is

$$\pi(H) = 1 - \frac{1}{\chi(H) - 1}.$$

***Remark* 1.5.2** (History)**.** Erdős and Stone (1946) proved this result when H is a complete multipartite graph. Erdős and Simonovits (1966) observed that the general case follows as a quick corollary. The proof given here is due to Erdős (1971).

***Example* 1.5.3.** When $H = K_{r+1}$, $\chi(H) = r + 1$, and so Theorem 1.5.1 agrees with Turán's theorem.

***Example* 1.5.4.** When H is the Petersen graph (shown in what follows), which has chromatic number 3, Theorem 1.5.1 tells us that $\operatorname{ex}(n, H) = (1/4 + o(1))n^2$. The Turán density of the Petersen graph is the same as that of a triangle, which may be somewhat surprising since the Petersen graph seems more complicated than the triangle.



It suffices to establish the Erdős–Stone–Simonovits theorem for complete r-partite graphs H, since every H with $\chi(H) = r$ is a subgraph of some complete r-partite graph.

Theorem 1.5.5 (Erdős–Stone theorem)
Fix $r \geq 2$ and $s \geq 1$. Let $H = K_{s,\ldots,s}$ be the complete r-partite graph with s vertices in each part. Then

$$\mathrm{ex}(n, H) = \left(1 - \frac{1}{r-1} + o(1)\right)\frac{n^2}{2}.$$

In other words, using the notation $K_r[s]$ for ***s*-blowup** of K_r, obtained by replacing each vertex of K_r by s duplicates of itself (so that $K_r[s] = H$ in the preceding theorem statement), the Erdős–Stone theorem says that

$$\pi(K_r[s]) = \pi(K_r) = 1 - \frac{1}{r-1},$$

In Section 1.3, we saw supersaturation (Theorem 1.3.4): when the edge density is significantly above the Turán density threshold $\pi(H)$, one finds not just a single copy of H but actually many copies. The Erdős–Stone theorem can be viewed in this light: above edge density $\pi(H)$, we find a large blowup of H.

The proof uses the following hypergraph extension of the KST theorem, which we will prove later in the section.

Recall the hypergraph Turán problem (Remark 1.3.3). Given an r-uniform hypergraph H (also known as an r-graph), we write $\mathrm{ex}(n, H)$ to be the maximum number of edges in an H-free r-graph.

The analogue of a complete bipartite graph for an r-graph is a complete r-partite r-graph $\boldsymbol{K^{(r)}_{s_1,\ldots,s_r}}$. Its vertex set consists of disjoint vertex parts $V_1, \ldots, V_r$ with $|V_i| = s_i$ for each i. Every r-tuple in $V_1 \times \cdots \times V_r$ is an edge.

Theorem 1.5.6 (Hypergraph KST)
For every fixed positive integers $r \geq 2$ and s,

$$\mathrm{ex}(n, K^{(r)}_{s,\ldots,s}) = o(n^r).$$

Proof of the Erdős–Stone theorem (Theorem 1.5.5). We already saw the lower bound to $\mathrm{ex}(n, H)$ using a Turán graph. It remains to prove an upper bound.

Let G be an H-free graph (where $H = K_{s,\ldots,s}$ is the complete r-partite graph in the theorem). Let $G^{(r)}$ be the r-graph with the same vertex set as G and whose edges are the r-cliques in G. Note that $G^{(r)}$ is $K^{(r)}_{s,\ldots,s}$-free, or else a copy of $K^{(r)}_{s,\ldots,s}$ in $G^{(r)}$ would be supported by a copy of H in G. Thus, by the hypergraph KST theorem (Theorem 1.5.6), $G^{(r)}$ has $o(n^r)$ edges. So G has $o(n^r)$ copies of K_r, and thus by the supersaturation theorem quoted above, the edge density of G is at most $\pi(K_r)+o(1)$, which equals $1-1/(r-1)+o(1)$ by Turán's theorem. □

In Section 2.6, we will give another proof of the Erdős–Stone–Simonovits theorem using the graph regularity method.

Hypergraph KST

To help keep notation simple, we first consider what happens for 3-uniform hypergraphs.

Theorem 1.5.7 (KST for 3-graphs)
For every s, there is some C such that

$$\mathrm{ex}(n, K^{(3)}_{s,s,s}) \le Cn^{3-1/s^2}.$$

Recall that the KST theorem (Theorem 1.4.2) was proved by counting the number of copies of $K_{s,1}$ in the graph in two different ways. For 3-graphs, we instead count the number of copies of $K^{(3)}_{s,1,1}$ in two different ways, one of which uses the KST theorem for $K_{s,s}$-free graphs.

Proof. Let G be a $K^{(3)}_{s,s,s}$-free 3-graph with n vertices and m edges. Let X denote the number of copies of $K^{(3)}_{s,1,1}$ in G. (When $s = 1$, we count each copy three times.)

Upper bound on X. Given a set S of s vertices, consider the set T of all unordered pairs of distinct vertices that would form a $K^{(3)}_{s,1,1}$ with S (i.e., every triple formed by combining a pair in T and a vertex of S is an edge of G). Note that T is the edge-set of a graph on the same n vertices. If T contains a $K_{s,s}$, then together with S we would have a $K^{(3)}_{s,s,s}$. Thus T is $K_{s,s}$-free, and hence by Theorem 1.4.2, $|T| = O_s(n^{2-1/s})$. Therefore,

$$X \lesssim_s \binom{n}{s} n^{2-1/s} \lesssim_s n^{s+2-1/s}.$$

Lower bound on X. We write $\deg(u,v)$ for the number of edges in G containing both u and v. Then, summing over all unordered pairs of distinct vertices u, v in G, we have

$$X = \sum_{u,v} \binom{\deg(u,v)}{s}.$$

As in the proof of Theorem 1.4.2, let

$$f_s(x) = \begin{cases} x(x-1)\cdots(x-s+1)/s! & \text{if } x \ge s-1, \\ 0 & x < s-1. \end{cases}$$

Then, f_s is convex and $f_s(x) = \binom{x}{s}$ for all nonnegative integers x. Since the average of $\deg(u,v)$ is $3m/\binom{n}{2}$,

$$X = \sum_{u,v} f_s(\deg(u,v)) \ge \binom{n}{2} f_s\left(\frac{3m}{\binom{n}{2}}\right).$$

Combining the upper and lower bounds, we have

$$\binom{n}{2}\left(\frac{3m}{\binom{n}{2}}\right)^s \lesssim_s n^{s+2-1/s}$$

and hence

$$m = O_s(n^{3-1/s^2}). \qquad \square$$

Exercise 1.5.8. Prove that $\operatorname{ex}(n, K^{(3)}_{r,s,t}) = O_{r,s,t}(n^{3-1/(rs)})$.

We can iterate further, using the same technique, to prove an analogous result for every uniformity, thereby giving us the statement (Theorem 1.5.6) used in our proof of the Erdős–Stone–Simonovits theorem earlier. Feel free to skip reading the next proof if you feel comfortable with generalizing the above proof to r-graphs.

Theorem 1.5.9 (Hypergraph KST)
For every $r \geq 2$ and $s \geq 1$, there is some C such that

$$\operatorname{ex}(n, K^{(r)}_{s,\dots,s}) \leq Cn^{r-s^{-r+1}},$$

where $K^{(r)}_{s,\dots,s}$ is the complete r-partite r-graph with s vertices in each of the r parts.

Proof. We prove by induction on r. The cases $r = 2$ and $r = 3$ were covered previously in Theorem 1.4.2 and Theorem 1.5.7. Assume that $r \geq 3$ and that the theorem has already been established for smaller values of r. (Actually we could have started at $r = 1$ if we adjust the definitions appropriately.)

Let G be a $K^{(r)}_{s,\dots,s}$-free r-graph with n vertices and m edges. Let X denote the number of copies of $K^{(r)}_{s,1,\dots,1}$ in G (when $s = 1$, we count each copy r times).

Upper bound on X. Given a set S of s vertices, consider the set T of all unordered $(r-1)$-tuples of vertices that would form a $K^{(r)}_{s,1,\dots,1}$ with S (where S is in one part, and the $r-1$ new vertices each in its own part). Note that T is the edge-set of an $(r-1)$ graph on the same n vertices. If T contains a $K^{(r-1)}_{s,\dots,s}$, then together with S we would have a $K^{(r)}_{s,\dots,s}$. Thus, T is $K^{(r-1)}_{s,\dots,s}$-free, and by the induction hypothesis, $|T| = O_{r,s}(n^{r-1-s^{-r+2}})$. Hence

$$X \lesssim_{r,s} \binom{n}{s} n^{r-1-s^{-r+2}} \lesssim_{r,s} n^{r+s-1-s^{-r+2}}.$$

Lower bound on X. Given a set U of vertices, we write $\deg U$ for the number of edges containing all vertices in U. Then

$$X = \sum_{U \in \binom{V(G)}{r-1}} \binom{\deg U}{s}.$$

Let $f_s(x)$ be defined as in the previous proof. Since the average of $\deg U$ over all $(r-1)$-element subsets U is $rm/\binom{n}{r-1}$, we have

$$X = \sum_{U \in \binom{V(G)}{r-1}} f_s(\deg U) \geq \binom{n}{r-1} f_s\left(\frac{rm}{\binom{n}{r-1}}\right).$$

Combining the upper and lower bounds, we have

$$\binom{n}{r-1} f_s\left(\frac{rm}{\binom{n}{r-1}}\right) \lesssim_{r,s} n^{s+r-1-s^{-r+2}}$$

and hence

$$m = O_{r,s}(n^{r-s^{-r+1}}). \qquad \square$$

Exercise 1.5.10 (Forbidding a multipartite complete hypergraph with unbalanced parts)**.** Prove that for every sequence of positive integers $s_1, \ldots, s_r$, there exists C so that

$$\mathrm{ex}(n, K^{(r)}_{s_1,\ldots,s_r}) \le Cn^{r-1/(s_1\cdots s_{r-1})}.$$

Exercise 1.5.11 (Erdős–Stone for hypergraphs)**.** Let H be an r-graph. Show that $\pi(H[s]) = \pi(H)$, where $H[s]$, the s-blowup of H, is obtained by replacing every vertex of H by s duplicates of itself.

1.6 Forbidding a Cycle

In this section, we consider the problem of determining $\mathrm{ex}(n, C_\ell)$, the maximum number of edges in an n-vertex graph without an ℓ-cycle.

Odd Cycles

First let us consider forbidding odd cycles. Let k be a positive integer. Then C_{2k+1} has chromatic number 3, and so the Erdős–Stone–Simonovits theorem (Theorem 1.5.1) tells us that

$$\mathrm{ex}(n, C_{2k+1}) = (1 + o(1))\frac{n^2}{4}.$$

In fact, an even stronger statement is true. If n is large enough (as a function of k), then the complete bipartite graph $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$ is always the extremal graph, just like in the triangle case.

Theorem 1.6.1 (Exact Turán number of an odd cycle)
Let k be a positive integer. Then for all sufficiently large integer n (i.e., $n \ge n_0(k)$ for some $n_0(k)$), one has

$$\mathrm{ex}(n, C_{2k+1}) = \left\lfloor \frac{n^2}{4} \right\rfloor.$$

We will not prove this theorem. See Füredi and Gunderson (2015) for a more recent proof.

More generally, Simonovits (1974) developed a stability method for exactly determining the Turán number of nonbipartite color-edge-critical graphs. Here we say that a graph if ***color-edge-critical*** if the removal of any edge strictly reduces the chromatic number.

Theorem 1.6.2 (Exact Turán number of a color-edge-critical graph)
Let F be a color-edge-critical graph with chromatic number $r + 1 \geq 3$. Then for all sufficiently large n (i.e., $n \geq n_0(F)$ for some $n_0(F)$), the Turán graph $T_{n,r}$ uniquely maximizes the number of edges among all n-vertex F-free graphs.

Forbidding Even Cycles

Let us now turn to forbidding even cycles. Since C_{2k} is bipartite, we know from the KST theorem that $\mathrm{ex}(n, C_{2k}) = o(n^2)$. The following upper bound was determined by Bondy and Simonovits (1974).

Theorem 1.6.3 (Even cycles)
For every integer $k \geq 2$, there exists a constant C so that

$$\mathrm{ex}(n, C_{2k}) \leq Cn^{1+1/k}.$$

***Remark* 1.6.4** (Tightness)**.** We will see in Section 1.10 a matching lower bound construction (up to constant factors) for $k = 2, 3, 5$. For all other values of k, it is an open question whether a matching lower bound construction exists.

Instead of proving the preceding theorem, we will prove a weaker result, stated in what follows. This weaker result has a short and neat proof, which hopefully gives some intuition as to why the above theorem should be true.

Theorem 1.6.5 (Short even cycles)
For any integer $k \geq 2$, there exists a constant C so that every graph G with n vertices and at least $Cn^{1+1/k}$ edges contains an even cycle of length at most $2k$.

In other words, Theorem 1.6.5 says that

$$\mathrm{ex}(n, \{C_4, C_6, \dots, C_{2k}\}) = O_k(n^{1+1/k}).$$

Here, given a set $\mathcal{F}$ of graphs, $\mathrm{ex}(n, \mathcal{F})$ denotes the maximum number of edges in an n-vertex graph that does not contain any graph in $\mathcal{F}$ as a subgraph.

To prove this theorem, we first clean up the graph by removing some edges and vertices to get a bipartite subgraph with large minimum degree.

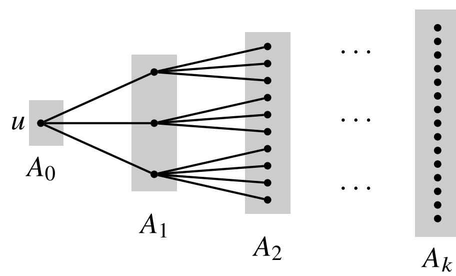
Lemma 1.6.6 (Large bipartite subgraph)
Every G has a bipartite subgraph with at least $e(G)/2$ edges.

Proof. Color every vertex with red or blue independently and uniformly at random. Then the expected number of nonmonochromatic edges is $e(G)/2$. Hence there exists a coloring that has at least $e(G)/2$ nonmonochromatic edges, and these edges form the desired bipartite subgraph. □

Lemma 1.6.7 (Large average degree implies subgraph with large minimum degree)
Let $t > 0$. Every graph with average degree $2t$ has a subgraph with minimum degree greater than t.

Proof. Let G be a graph with average degree $2t$. Removing a vertex of degree at most t cannot decrease the average degree, since the total degree goes down by at most $2t$ and so the post-deletion graph has average degree of at least $(2e(G) - 2t)/(v(G) - 1)$, which is at least $2e(G)/v(G)$ since $2e(G)/v(G) \geq 2t$. Let us repeatedly delete vertices of degree at most t in the remaining graph until every vertex has degree more than t. This algorithm must terminate with a nonempty graph since we cannot ever drop below $2t$ vertices in this process (as such a graph would have average degree less than $2t$). □

Proof of Theorem 1.6.5. The idea is to use a **breath-first search**. Suppose G contains no even cycles of length at most $2k$. Applying Lemma 1.6.6 followed by Lemma 1.6.7, we find a bipartite subgraph G' of G with minimum degree $> t := e(G)/(2v(G))$. Let u be an arbitrary vertex of G'. For each $i = 0, 1, \ldots, k$, let A_i denote the set of vertices at distance exactly i from u.



For each $i = 1, \ldots, k - 1$, every vertex of A_i has

- no neighbors inside A_i (or else G' would not be bipartite),
- exactly one neighbor in A_{i-1} (else we can backtrace through two neighbors, which must converge at some point to form an even cycle of length at most $2k$),
- and thus $> t - 1$ neighbors in A_{i+1} (by the minimum degree assumption on G').

Therefore, each layer A_i expands to the next by a factor of at least $t - 1$. Hence

$$v(G) \geq |A_k| \geq (t-1)^k \geq \left(\frac{e(G)}{2v(G)} - 1\right)^k$$

and thus

$$e(G) \leq 2v(G)^{1+1/k} + 2v(G). \qquad \square$$

Exercise 1.6.8 (Extremal number of trees)**.** Let T be a tree with k edges. Show that $\mathrm{ex}(n, T) \leq kn$.

1.7 Forbidding a Sparse Bipartite Graph: Dependent Random Choice

Every bipartite graph H is contained in some $K_{s,t}$, and thus by the KST theorem (Theorem 1.4.2), $\mathrm{ex}(n, H) \leq \mathrm{ex}(n, K_{s,t}) = O_{s,t}(n^{2-1/s})$. The main result of this section gives a significant improvement when the maximum degree of H is small. The proof introduces an important probabilistic technique known as **dependent random choice**.

Theorem 1.7.1 (Bounded degree bipartite graph: Turán number upper bound)
Let H be a bipartite graph with vertex bipartition $A \cup B$ such that every vertex in A has degree at most r. Then there exists a constant $C = C_H$ such that for all n,

$$\mathrm{ex}(n, H) \leq Cn^{2-1/r}.$$

***Remark* 1.7.2** (History)**.** The result was first proved by Füredi (1991). The proof given here is due to Alon, Krivelevich, and Sudakov (2003a). For more applications of the dependent random choice technique see the survey by Fox and Sudakov (2011).

***Remark* 1.7.3** (Tightness)**.** The exponent $2 - 1/r$ is best possible as a function of r. Indeed, we will see in the following section that for every r there exists some s so that $\mathrm{ex}(n, K_{r,s}) \geq cn^{2-1/r}$ for some $c = c(r, s) > 0$.

On the other hand, for specific graphs G, Theorem 1.7.1 may not be tight. For example, $\mathrm{ex}(n, C_6) = \Theta(n^{4/3})$, whereas Theorem 1.7.1 only tells us that $\mathrm{ex}(n, C_6) = O(n^{3/2})$.

Given a graph G with many edges, we wish to find a large subset U of vertices such that every r-vertex subset of U has many common neighbors in G. (Even the case $r = 2$ is interesting.) Once such a U is found, we can then embed the B-vertices of H into U. It will then be easy to embed the vertices of A. The tricky part is to find such a U.

***Remark* 1.7.4** (Intuition)**.** We want to host a party so that each pair of partygoers has many common friends. (Here G is the friendship graph.) Whom should we invite? Inviting people uniformly at random is not a good idea. (Why?) Perhaps we can pick some random individual (Alice) to host a party inviting all her friends. Alice's friends are expected to share some common friends – at least they all know Alice.

We can take a step further, and pick a few people at random (Alice, Bob, Carol, David) and have them host a party and invite all their common friends. This will likely be an even more sociable crowd. At least all the party goers will know all the hosts, and likely even more. As long as the social network is not too sparse, there should be lots of invitees.

Some invitees (e.g., Zack) might feel a bit out of place at the party – maybe they don't have many common friends with other partygoers. (They all know the hosts but maybe Zack doesn't know many others.) To prevent such awkwardness, the hosts will cancel Zack's invitation. There shouldn't be too many people like Zack. The party must go on.

Here is the technical statement that we will prove. While there are many parameters, the specific details are less important compared to the proof technique. This is quite a tricky proof.

Theorem 1.7.5 (Dependent random choice)
Let n, r, m, t be positive integers and $\alpha > 0$. Then every graph G with n vertices and at least $\alpha n^2/2$ edges contains a vertex subset U with

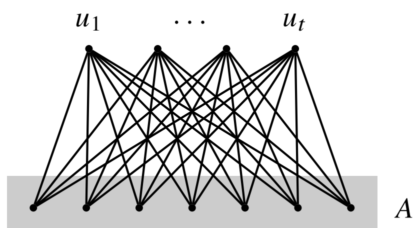
$$|U| \geq n\alpha^t - \binom{n}{r}\left(\frac{m}{n}\right)^t$$

such that every r-element subset S of U has more than m common neighbors in G.

***Remark* 1.7.6** (Parameters). In the theorem statement, t is an auxiliary parameter that does not appear in the conclusion. While one can optimize for t, it is instructive and convenient to leave it as is. The theorem is generally applied to graphs with at least n^{2-c} edges, for some small $c > 0$, and we can play with the parameters to get $|U|$ and m both large as desired.

Proof. We say that an r-element subset of $V(G)$ is "bad" if it has at most m common neighbors in G.

Let $u_1, \ldots, u_t$ be vertices chosen uniformly and independently at random from $V(G)$. (These vertices are chosen "with replacement," i.e., they can repeat.) Let A be their common neighborhood. (Keep in mind that $u_1, \ldots, u_t, A$ are random. It may be a bit confusing in this proof to see what is random and what is not.)



Each fixed vertex $v \in V(G)$ has probability $(\deg(v)/n)^t$ of being adjacent to all of $u_1, \ldots, u_t$, and so, by linearity of expectations and convexity,

$$\mathbb{E}\,|A| = \sum_{v \in V(G)} \mathbb{P}(v \in A) = \sum_{v \in V(G)} \left(\frac{\deg(v)}{n}\right)^t \geq n \left(\frac{1}{n} \sum_{v \in V} \frac{\deg(v)}{n}\right)^t \geq n\alpha^t.$$

For any fixed $R \subseteq V(G)$,

$$\mathbb{P}(R \subseteq A) = \mathbb{P}(R \text{ is complete to } u_1, \ldots, u_t) = \left(\frac{\text{\# common neighbors of } R}{n}\right)^t.$$

If R is a bad r-vertex subset, then it has at most m common neighbors, and so

$$\mathbb{P}(R \subseteq A) \leq \left(\frac{m}{n}\right)^t.$$

Therefore, summing over all $\binom{n}{r}$ possible r-vertex subsets $R \subseteq V(G)$, by linearity of expectation,

$$\mathbb{E}[\text{the number of bad } r\text{-vertex subsets of } A] \leq \binom{n}{r}\left(\frac{m}{n}\right)^t.$$

Let U be obtained from A by deleting an element from each bad r-vertex subset. So, U has no bad r-vertex subsets. Also,

$$\begin{aligned}\mathbb{E}\,|U| &\geq \mathbb{E}\,|A| - \mathbb{E}[\text{the number of bad } r\text{-vertex subsets of } A] \\ &\geq n\alpha^t - \binom{n}{r}\left(\frac{m}{n}\right)^t.\end{aligned}$$

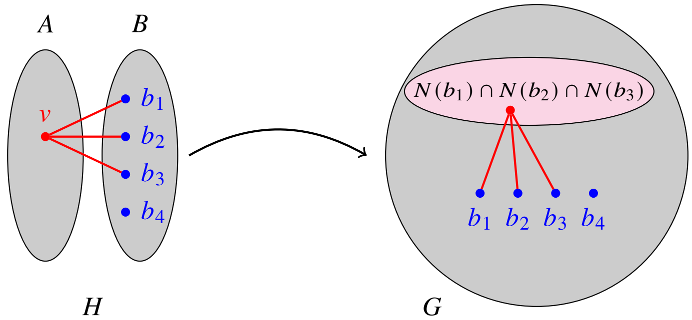
Thus, there exists some U with at least this size, with the property that all its r-vertex subsets have more than m common neighbors. $\square$

Now we are ready to show Theorem 1.7.1, which we recall says that, for a bipartite graph H with vertex bipartition $A \cup B$ such that every vertex in A has degree at most r, one has $\operatorname{ex}(n, H) = O_H(n^{2-1/r})$.

Proof of Theorem 1.7.1. Let G be a graph with n vertices and at least $Cn^{2-1/r}$ edges. By choosing C large enough (depending only on $|A| + |B|$), we have

$$n\left(2Cn^{-1/r}\right)^r - \binom{n}{r}\left(\frac{|A| + |B|}{n}\right)^r \geq |B|\,.$$

We want to show that G contains H as a subgraph. By dependent random choice (Theorem 1.7.5 applied with $t = r$), we can embed the B-vertices of H into G so that every r-vertex subset of B (now viewed as a subset of $V(G)$) has $> |A| + |B|$ common neighbors.



Next, we embed the vertices of A one at a time. Suppose we need to embed $v \in A$ (some previous vertices of A may have already been embedded at this point). Note that v has $\leq r$ neighbors in B, and these $\leq r$ vertices in B have $> |A| + |B|$ common neighbors in G. While some of these common neighbors may have already been used up in earlier steps to embed vertices of H, there are enough of them that they cannot all be used up, and thus we can embed v to some remaining common neighbor. This process ends with an embedding of H into G. $\square$

Exercise 1.7.7. Let H be a bipartite graph with vertex bipartition $A \cup B$, such that r vertices in A are complete to B, and all remaining vertices in A have degree at most r. Prove that there is some constant $C = C_H$ such that $\operatorname{ex}(n, H) \leq Cn^{2-1/r}$ for all n.

Exercise 1.7.8. Let $\varepsilon > 0$. Show that, for sufficiently large n, every K_4-free graph with n vertices and at least εn^2 edges contains an independent set of size at least $n^{1-\varepsilon}$.

Exercise 1.7.9 (Extremal numbers of degenerate graphs)**.**

(a*) Prove that there is some absolute constant $c > 0$ so that for every positive integer r, every n-vertex graph with at least $n^{2-c/r}$ edges contains disjoint nonempty vertex subsets A and B such that every subset of at most r vertices in A has at least n^c common neighbors in B and every subset of at most r vertices in B has at least n^c neighbors in A.

Hint: Apply the technique from the dependent random choice proof repeatedly back and forth to the two vertex parts.

(b) We say that a graph H is ***r-degenerate*** if its vertices can be ordered so that every vertex has at most r neighbors that appear before it in the ordering. Show that for every r-degenerate bipartite graph H there is some constant $C > 0$ so that $\mathrm{ex}(n, H) \le Cn^{2-c/r}$, where c is the same absolute constant from part (a). (Here c should not depend on H or r.)

1.8 Lower Bound Constructions: Overview

We proved various upper bounds on $\mathrm{ex}(n, H)$ in earlier sections. When H is nonbipartite, the Turán graph construction (Construction 1.2.2) shows that the upper bound in the Erdős–Stone–Simonovits theorem (Theorem 1.5.1) is tight up to lower-order terms. However, when H is bipartite, so that $\mathrm{ex}(n, H) = o(n^2)$, we have not seen any nontrivial lower bound constructions. In the remainder of this chapter, we will see some methods for constructing H-free graphs for bipartite H. In some cases, these constructions will have enough edges to match the upper bounds on $\mathrm{ex}(n, H)$ from earlier sections. However, for most bipartite graphs H, there is a gap in known upper and lower bounds on $\mathrm{ex}(n, H)$. It is a central problem in extremal graph theory to close this gap.

We will see three methods for constructing H-free graphs.

Randomized Constructions

The idea is to take a random graph at a density that gives a small number of copies of H, and then destroy these copies of H by removing some edges from the random graph. The resulting graph is then H-free. This method is easy to implement and applies quite generally to all H. For example, it will be shown that

$$\mathrm{ex}(n, H) = \Omega_H\left(n^{2-\frac{v(H)-2}{e(H)-1}}\right).$$

However, bounds arising from this method are usually not tight.

Algebraic Constructions

The idea is to use algebraic geometry over a finite field to construct a graph. Its vertices correspond to geometric objects such as points or lines. Its edges correspond to incidences or other algebraic relations. These constructions sometimes give tight bounds. They work for a small number of graphs H, and usually require a different ad hoc idea for each H. They work rarely, but when they do, they can appear quite mysterious, or even magical. Many important tight lower bounds on bipartite extremal numbers arise this way. In particular it will be shown that

$$\mathrm{ex}(n, K_{s,t}) = \Omega_{s,t}\left(n^{2-1/s}\right) \qquad \text{whenever } t \geq (s-1)! + 1,$$

thereby matching the KST theorem (Theorem 1.4.2) for such s, t. Also, it will be shown that

$$\mathrm{ex}(n, C_{2k}) = \Omega_k\left(n^{1+1/k}\right) \qquad \text{whenever } k \in \{2, 3, 5\},$$

thereby matching Theorem 1.6.3 for these values of k.

Randomized Algebraic Constructions

In algebraic constructions, usually we specify the edges using some specific well-chosen polynomials. A powerful recent idea is to choose the edge-defining polynomials at random.

1.9 Randomized Constructions

We use the probabilistic method to construct an H-free graph. The Erdős–Rényi random graph $\mathbf{G}(n, p)$ is the random graph on n vertices where every pair of vertices forms an edge independently with probability p. We first take a $\mathbf{G}(n, p)$ with an appropriately chosen p. The number of copies of H in $\mathbf{G}(n, p)$ is expected to be small, and we can destroy all such copies of H from the random graph by removing some edges. The remaining graph will then be H-free.

The method of starting with a simple random object and then modifying it is sometimes called **alteration method** or the **deletion method**.

Theorem 1.9.1 (Randomized lower bound)
Let H be a graph with at least two edges. Then there exists a constant $c = c_H > 0$, so that for all $n \geq 2$, there exists an H-free graph on n vertices with at least $cn^{2-\frac{v(H)-2}{e(H)-1}}$ edges. In other words,

$$\mathrm{ex}(n, H) \geq cn^{2-\frac{v(H)-2}{e(H)-1}}.$$

Proof. Let G be an instance of the Erdős–Rényi random graph $\mathbf{G}(n, p)$, with

$$p = \frac{1}{4}n^{-\frac{v(H)-2}{e(H)-1}}$$

(chosen with hindsight). We have $\mathbb{E}\, e(G) = p\binom{n}{2}$. Let X denote the number of copies of H in G. Then, our choice of p ensures that

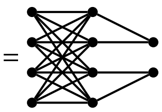
$$\mathbb{E}X \leq p^{e(H)} n^{v(H)} \leq \frac{p}{2}\binom{n}{2} = \frac{1}{2}\mathbb{E}\, e(G).$$

Thus

$$\mathbb{E}[e(G) - X] \geq \frac{p}{2}\binom{n}{2} \gtrsim n^{2-\frac{v(H)-2}{e(H)-1}}.$$

Take a graph G such that $e(G) - X$ is at least its expectation. Remove one edge from each copy of H in G, and we get an H-free graph with at least $e(G) - X \gtrsim n^{2-\frac{v(H)-2}{e(H)-1}}$ edges. $\square$

For some graphs H, we can bootstrap Theorem 1.9.1 to give an even better lower bound. For example, if

$H =$  ,

then $v(H) = 10$ and $e(H) = 20$, so applying Theorem 1.9.1 directly gives

$$\mathrm{ex}(n, H) \gtrsim n^{2-8/19}.$$

On the other hand, any $K_{4,4}$-free graph is automatically H-free. Applying Theorem 1.9.1 to $K_{4,4}$ (8-vertex 16-edge) actually gives a better lower bound $(2 - 6/15 > 2 - 8/19)$:

$$\mathrm{ex}(n, H) \geq \mathrm{ex}(n, K_{4,4}) \gtrsim n^{2-6/15}.$$

In general, given H, we should apply Theorem 1.9.1 to the subgraph of H with the maximum $(e(H) - 1)/(v(H) - 2)$ ratio. This gives the following corollary, which sometimes gives a better lower bound than directly applying Theorem 1.9.1.

Definition 1.9.2 (2-density)
The ***2-density*** of a graph H is defined by

$$\boldsymbol{m_2(H)} := \max_{\substack{H' \subseteq H \\ e(H') \geq 2}} \frac{e(H') - 1}{v(H') - 2}.$$

Corollary 1.9.3 (Randomized lower bound)
For any graph H with at least two edges, there exists constant $c = c_H > 0$ such that

$$\mathrm{ex}(n, H) \geq cn^{2-1/m_2(H)}.$$

Proof. Let H' be the subgraph of H with $m_2(H) = \frac{e(H')-1}{v(H')-2}$. Then $\mathrm{ex}(n, H) \geq \mathrm{ex}(n, H')$, and we can apply Theorem 1.9.1 to get $\mathrm{ex}(n, H) \geq cn^{2-1/m_2(H)}$. □

***Example* 1.9.4.** Theorem 1.9.1 combined with the upper bound from the KST theorem (Theorem 1.4.2) gives that for every fixed $2 \leq s \leq t$,

$$n^{2-\frac{s+t-2}{st-1}} \lesssim \mathrm{ex}(n, K_{s,t}) \lesssim n^{2-\frac{1}{s}}.$$

When t is large compared to s, the exponents in the preceding two bounds are close to each other (but never equal). When $t = s$, the preceding bounds specialize to

$$n^{2-\frac{2}{s+1}} \lesssim \mathrm{ex}(n, K_{s,s}) \lesssim n^{2-\frac{1}{s}}.$$

In particular, for $s = 2$,

$$n^{4/3} \lesssim \mathrm{ex}(n, K_{2,2}) \lesssim n^{3/2}.$$

It turns out that the upper bound is tight. We will show this in the next section using an algebraic construction.

Exercise 1.9.5. Show that if H is a bipartite graph containing a cycle of length $2k$, then $\operatorname{ex}(n,H) \gtrsim_H n^{1+1/(2k-1)}$.

Exercise 1.9.6. Find a graph H with $\chi(H) = 3$ and $\operatorname{ex}(n,H) > \frac{1}{4}n^2 + n^{1.99}$ for all sufficiently large n.

1.10 Algebraic Constructions

In this section, we use algebraic methods to construct $K_{s,t}$-free graphs for certain values of (s,t), as well as C_{2k}-free graphs for certain values of k. In both cases, the constructions are optimal in that they match the upper bounds up to a constant factor.

$K_{2,2}$-free

We begin by constructing $K_{2,2}$-free graphs with the number of edges matching the KST theorem. The construction is due to Erdős, Rényi, and Sós (1966) and Brown (1966) independently.

Theorem 1.10.1 (Construction of $K_{2,2}$-free graphs)

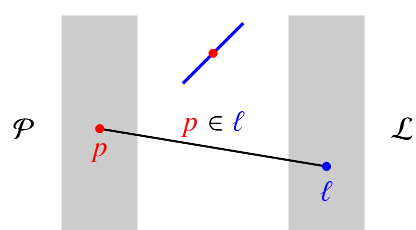
$$\operatorname{ex}(n, K_{2,2}) \geq \left(\frac{1}{2} - o(1)\right) n^{3/2}.$$

Combining with the KST theorem, we obtain the corollary.

Corollary 1.10.2 (Turán number of $K_{2,2}$)

$$\operatorname{ex}(n, K_{2,2}) = \left(\frac{1}{2} - o(1)\right) n^{3/2}.$$

Before giving the proof of Theorem 1.10.1, let us first sketch the geometric intuition. Given a set of points $\mathcal{P}$ and a set of lines $\mathcal{L}$, the ***point-line incidence graph*** is the bipartite graph with two vertex parts $\mathcal{P}$ and $\mathcal{L}$, where $p \in \mathcal{P}$ and $\ell \in \mathcal{L}$ are adjacent if $p \in \ell$.



A point-line incidence graph is C_4-free. Indeed, a C_4 would correspond to two lines both passing through two distinct points, which is impossible.

We want to construct a set of points and a set of lines so that there are many incidences. To do this, we take all points and all lines in a finite field plane $\mathbb{F}_p^2$. There are p^2 points and $p^2 + p$ lines. Since every line contains p points, the graph has around p^3 edges, and so $\mathrm{ex}(2p^2 + p, K_{2,2}) \geq p^3$. By rounding down an integer n to the closest number of the form $2p^2 + p$ for a prime p, we already see that $\mathrm{ex}(n, K_{2,2}) \gtrsim n^{3/2}$ for all n. Here we use a theorem from number theory regarding large gaps in primes, which we quote in what follows without proof. (This strategy does not work if we instead take points and lines in $\mathbb{R}^2$; see the Szemerédi–Trotter theorem in Section 8.2).

Theorem 1.10.3 (Large gaps between primes)
The largest prime below N has size $N - o(N)$.

***Remark* 1.10.4** (Large gaps between primes). The preceding result already follows from the prime number theorem, which says that the number of primes up to N is $(1+o(1))N/\log N$. The best quantitative result, due to Baker, Harman, and Pintz (2001), says that there exists a prime in $[N - N^{0.525}, N]$ for all sufficiently large N. Cramer's conjecture, which is wide open and based on a random model of the primes, speculates that the $o(N)$ in Theorem 1.10.3 may be replaced by $O((\log N)^2)$. An easier claim is Bertrand's postulate, which says that there is a prime between N and $2N$ for every N, and this already suffices for proving $\mathrm{ex}(n, K_{2,2}) \gtrsim n^{3/2}$.

To get a better constant in the preceding construction, we optimize somewhat by using the same vertices to represent both points and lines. This pairing of points and lines is known as ***polarity*** in projective geometry, and this construction is known as the ***polarity graph***. (This usually refers to the projective plane version of the construction.)

Proof of Theorem 1.10.1. Let p denote the largest prime such that $p^2 - 1 \leq n$. Then $p = (1 - o(1))\sqrt{n}$ by Theorem 1.10.3. Let G be a graph with vertex set $V(G) = \mathbb{F}_p^2 \setminus \{(0, 0)\}$ and an edge between (x, y) and (a, b) if and only if $ax + by = 1$ in $\mathbb{F}_p$.

For any two distinct vertices (a, b) and (a', b') in $V(G)$, they have at most one common neighbor since there is at most one solution to the system $ax + by = 1$ and $a'x + b'y = 1$. Therefore, G is $K_{2,2}$-free. (This is where we use the fact that two lines intersect in at most one point.)

For every $(a, b) \in V(G)$, there are exactly p vertices (x, y) satisfying $ax + by = 1$. However, one of those vertices could be (a, b) itself. So every vertex in G has degree p or $p - 1$. Hence G has at least $(p^2 - 1)(p - 1)/2 = (1/2 - o(1))n^{3/2}$ edges. □

$K_{3,3}$-*free*

Next, we construct $K_{3,3}$-free graphs with the number of edges matching the KST theorem. This construction is due to Brown (1966).

Theorem 1.10.5 (Construction of $K_{3,3}$-free graphs)
For every n,

$$\mathrm{ex}(n, K_{3,3}) \geq \left(\frac{1}{2} - o(1)\right) n^{5/3}.$$

Consider the incidences between points in $\mathbb{R}^3$ and unit spheres. This graph is $K_{3,3}$-free since no three unit spheres can share three distinct common points. Again, one needs to do this over a finite field to attain the desired bounds, but it is easier to visualize the setup in Euclidean space, where it is clearly true.

Proof sketch. Let p be the largest prime less than $n^{1/3}$. Fix a nonzero element $d \in \mathbb{F}_p$, which we take to be a quadratic residue if $p \equiv 3 \pmod 4$ and a quadratic nonresidue if $p \not\equiv 3 \pmod 4$. Construct a graph G with vertex set $V(G) = \mathbb{F}_p^3$, and an edge between (x, y, z) and $(a, b, c) \in V(G)$ if and only if

$$(a - x)^2 + (b - y)^2 + (c - z)^2 = d.$$

It turns out that each vertex has $(1 - o(1))p^2$ neighbors. (The intuition here is that, for a fixed (a, b, c), if we choose $x, y, z \in \mathbb{F}_p$ independently and uniformly at random, then the resulting sum $(a - x)^2 + (b - y)^2 + (c - z)^2$ is roughly uniformly distributed, and hence equals to d with probability close to $1/p$.) It remains to show that the graph is $K_{3,3}$-free. To see this, think about how one might prove this claim in $\mathbb{R}^3$ via algebraic manipulations. We compute the radical planes between pairs of spheres as well as the intersections of these radical planes (i.e., the radical axis). The claim boils down to the fact that no sphere has three collinear points, which is true due to the quadratic (non)residue hypothesis on d. The details are omitted.

Thus G is a $K_{3,3}$-free graph on $p^3 \le n$ vertices and with at least $(1/2 - o(1))p^5 = (1/2 - o(1))n^{5/3}$ edges. □

It is unknown if the preceding ideas can be extended to construct $K_{4,4}$-free graphs with $\Omega(n^{7/4})$ edges. It is a major open problem to determine the asymptotics of $\mathrm{ex}(n, K_{4,4})$.

Conjecture 1.10.6 (KST theorem is tight)
For every fixed $s \ge 4$, one has

$$\mathrm{ex}(n, K_{s,s}) = \Theta_s(n^{2-1/s}).$$

$K_{s,t}$-free

Now we present a substantial generalization of the above constructions, due to Kollár, Rónyai, and Szabó (1996) and Alon, Rónyai, and Szabó (1999). It gives a matching lower bound (up to a constant factor) to the KST theorem for $K_{s,t}$ whenever t is sufficiently large compared to s.

Theorem 1.10.7 (Tightness of KST bound when $t > (s-1)!$)
Fix a positive integer $s \ge 2$. Then

$$\mathrm{ex}(n, K_{s,(s-1)!+1}) \ge \left(\frac{1}{2} - o(1)\right) n^{2-1/s}.$$

Corollary 1.10.8 (Tightness of KST bound when $t > (s-1)!$)
If $t > (s-1)!$, then

$$\mathrm{ex}(n, K_{s,t}) = \Theta_{s,t}(n^{2-1/s}).$$

We first prove a slightly weaker version of Theorem 1.10.7, namely that

$$\operatorname{ex}(n, K_{s,s!+1}) \geq \left(\frac{1}{2} - o(1)\right) n^{2-1/s}.$$

(Kollár, Rónyai, and Szabó 1996). Afterwards, we will modify the construction to prove Theorem 1.10.7.

Let p be a prime. Recall that the ***norm map*** $N \colon \mathbb{F}_{p^s} \to \mathbb{F}_p$ is defined by

$$N(x) := x \cdot x^p \cdot x^{p^2} \cdots x^{p^{s-1}} = x^{\frac{p^s-1}{p-1}}.$$

Note that $N(x) \in \mathbb{F}_p$ for all $x \in \mathbb{F}_{p^s}$ since $N(x)^p = N(x)$ and $\mathbb{F}_p$ is the set of elements in $\mathbb{F}_{p^s}$ invariant under the automorphism $x \mapsto x^p$. Furthermore, since $\mathbb{F}_{p^s}^\times$ is a cyclic group of order $p^s - 1$, we know that

$$\left|\{x \in \mathbb{F}_{p^s} : N(x) = 1\}\right| = \frac{p^s-1}{p-1}. \tag{1.2}$$

Construction 1.10.9 (Norm graph)

NormGraph$_{p,s}$ is defined to be the graph with vertex set $\mathbb{F}_{p^s}$ and an edge between distinct $a, b \in \mathbb{F}_{p^s}$ if $N(a+b) = 1$.

By (1.2), every vertex in $\mathsf{NormGraph}_{p,s}$ has degree at least

$$\frac{p^s-1}{p-1} - 1 \geq p^{s-1}.$$

(We had to subtract 1 in case $N(x+x) = 1$.) And thus the number of edges is at least $p^{2s-1}/2$. It remains to establish that $\mathsf{NormGraph}_{p,s}$ is $K_{s,s!+1}$-free. Once this is done, we can take p to be the largest prime at most $n^{1/s}$, and then

$$\operatorname{ex}(n, K_{s,s!+1}) \geq \operatorname{ex}(p^s, K_{s,s!+1}) \geq \frac{p^{2s-1}}{2} \geq \left(\frac{1}{2} - o(1)\right) n^{2-1/s}.$$

Proposition 1.10.10

$\mathsf{NormGraph}_{p,s}$ is $K_{s,s!+1}$-free for all $s \geq 2$.

We wish to upper bound the number of common neighbors to a set of s vertices. This amount to showing that a certain system of algebraic equations cannot have too many solutions. We quote without proof the following key algebraic result from Kollár, Rónyai, and Szabó (1996), which can be proved using algebraic geometry.

Theorem 1.10.11

Let $\mathbb{F}$ be any field and $a_{ij}, b_i \in \mathbb{F}$ such that $a_{ij} \neq a_{i'j}$ for all $i \neq i'$. Then the system of equations

$$\begin{aligned}(x_1 - a_{11})(x_2 - a_{12}) \cdots (x_s - a_{1s}) &= b_1 \\ (x_1 - a_{21})(x_2 - a_{22}) \cdots (x_s - a_{2s}) &= b_2 \\ &\vdots \\ (x_1 - a_{s1})(x_2 - a_{s2}) \cdots (x_s - a_{ss}) &= b_s\end{aligned}$$

has at most $s!$ solutions $(x_1, \ldots, x_s) \in \mathbb{F}^s$.

***Remark* 1.10.12** (Special case $b = 0$)**.** Consider the special case when all the b_i are 0. In this case, since the a_{ij} are distinct for each fixed j, every solution to the system corresponds to a permutation $\pi \colon [s] \to [s]$, setting $x_i = a_{i\pi(i)}$. So there are exactly $s!$ solutions in this special case. The difficult part of the theorem says that the number of solutions cannot increase if we move b away from the origin.

Proof of Proposition 1.10.10. Consider distinct $y_1, y_2, \ldots, y_s \in \mathbb{F}_{p^s}$. We wish to bound the number of common neighbors x. Recall that in a field with characteristic p, we have the identity $(x + y)^p = x^p + y^p$ for all x, y. So,

$$\begin{aligned}1 = N(x + y_i) &= (x + y_i)(x + y_i)^p \ldots (x + y_i)^{p^{s-1}} \\ &= (x + y_i)(x^p + y_i^p) \ldots (x^{p^{s-1}} + y_i^{p^{s-1}})\end{aligned}$$

for all $1 \leq i \leq s$. By Theorem 1.10.11, these s equations (as i ranges over $[s]$) have at most $s!$ solutions in x. Note the hypothesis of Theorem 1.10.11 is satisfied since $y_i^p = y_j^p$ if and only if $y_i = y_j$ in $\mathbb{F}_{p^s}$. $\square$

Now we modify the norm graph construction to forbid $K_{s,(s-1)!+1}$, thereby yielding Theorem 1.10.7.

Construction 1.10.13 (Projective norm graph)

Let **ProjNormGraph**$_{p,s}$ be the graph with vertex set $\mathbb{F}_{p^{s-1}} \times \mathbb{F}_p^\times$, where two vertices $(X, x), (Y, y) \in \mathbb{F}_{p^{s-1}} \times \mathbb{F}_p^\times$ are adjacent if and only if

$$N(X + Y) = xy.$$

In ProjNormGraph$_{p,s}$, every vertex (X, x) has degree $p^{s-1} - 1$ since its neighbors are $(Y, N(X + Y)/x)$ for all $Y \neq -X$. There are $(p^{s-1} - 1)p^{s-1}(p - 1)/2$ edges. As earlier, it remains to show that this graph is $K_{s,(s-1)!+1}$-free. Once we know this, by taking p to be the largest prime satisfying $p^{s-1}(p - 1) \leq n$, we obtain the desired lower bound

$$\operatorname{ex}(n, K_{s,(s-1)!+1}) \geq \frac{1}{2}(p^{s-1} - 1)p^{s-1}(p - 1) \geq \left(\frac{1}{2} - o(1)\right) n^{2-1/s}.$$

Proposition 1.10.14

ProjNormGraph$_{p,s}$ is $K_{s,(s-1)!+1}$-free.

Proof. Fix distinct $(Y_1, y_1), \dots, (Y_s, y_s) \in \mathbb{F}_{p^{s-1}} \times \mathbb{F}_p^\times$. We wish to show that there are at most $(s-1)!$ solutions $(X, x) \in \mathbb{F}_{p^{s-1}} \times \mathbb{F}_p^\times$ to the system of equations

$$N(X + Y_i) = xy_i, \qquad i = 1, \dots, s.$$

Assume this system has at least one solution. Then if $Y_i = Y_j$ with $i \neq j$ we must have that $y_i = y_j$. Therefore all the Y_i are distinct. For each $i < s$, dividing $N(X + Y_i) = xy_i$ by $N(X + Y_s) = xy_s$ gives

$$N\left(\frac{X + Y_i}{X + Y_s}\right) = \frac{y_i}{y_s}, \qquad i = 1, \dots, s-1.$$

Dividing both sides by $N(Y_i - Y_s)$ gives

$$N\left(\frac{1}{X + Y_s} + \frac{1}{Y_i - Y_s}\right) = \frac{y_i}{N(Y_i - Y_s) y_s}, \qquad i = 1, \dots, s-1.$$

Now apply Theorem 1.10.11 (same as in the proof of Proposition 1.10.10). We deduce that there are at most $(s-1)!$ choices for X, and each such X automatically determines $x = N(X + Y_1)/y_1$. Thus, there are at most $(s-1)!$ solutions (X, x). $\square$

C_4, C_6, C_{10}-*free*

Finally, let us turn to constructions of C_{2k}-**free graphs**. We had mentioned in Section 1.6 that $\operatorname{ex}(C_{2k}, n) = O_k(n^{1+1/k})$. We saw a matching lower bound construction for 4-cycles. Now we give matching constructions for 6-cycles and 10-cycles. (It remains an open problem for other cycle lengths.)

Theorem 1.10.15 (Tight lower bound for avoiding C_{2k} for $k \in \{2, 3, 5\}$)
Let $k \in \{2, 3, 5\}$. Then there is a constant $c > 0$ such that for every n,

$$\operatorname{ex}(n, C_{2k}) \geq cn^{1+1/k}.$$

***Remark* 1.10.16** (History)*.* The existence of such C_{2k}-free graphs for $k \in \{3, 5\}$ is due to Benson (1966) and Singleton (1966). The construction given here is due to Wenger (1991), with a simplified description due to Conlon (2021).

The following construction generalizes the point-line incidence graph construction earlier for the C_4-free graph in Theorem 1.10.1. Here we consider a special set of lines in $\mathbb{F}_q^k$, whereas previously for C_4 we took all lines in $\mathbb{F}_q^2$.

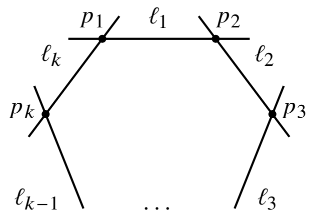
Construction 1.10.17 (C_{2k}-free construction for $k \in \{2, 3, 5\}$)
Let q be a prime power. Let $\mathcal{L}$ denote the set of all lines in $\mathbb{F}_q^k$ whose direction can be written as $(1, t, \dots, t^{k-1})$ for some $t \in \mathbb{F}_q$. Let $G_{q,k}$ denote the bipartite point-line incidence graph with vertex sets $\mathbb{F}_q^k$ and $\mathcal{L}$. That is, $(p, \ell) \in \mathbb{F}_q^k \times \mathcal{L}$ is an edge if and only if $p \in \ell$.

We have $|\mathcal{L}| = q^k$, since to specify a line in $\mathcal{L}$ we can provide a point with first coordinate equal to zero, along with a choice of $t \in \mathbb{F}_q$ giving the direction of the line. So the graph $G_{q,k}$

has $n = 2q^k$ vertices. Since each line contains exactly q points, there are exactly $q^{k+1} \asymp n^{1+1/k}$ edges in the graph. It remains to show that this graph is C_{2k}-free whenever $k \in \{2, 3, 5\}$. Then Theorem 1.10.15 would follow after the usual trick of taking q to be the largest prime with $2q^k < n$.

Proposition 1.10.18
Let $k \in \{2, 3, 5\}$. The graph $G_{q,k}$ from Construction 1.10.17 is C_{2k}-free.

Proof. A $2k$-cycle in $G_{q,k}$ would correspond to $p_1, \ell_1, \ldots, p_k, \ell_k$ with distinct $p_1, \ldots, p_k \in \mathbb{F}_q^k$ and distinct $\ell_1, \ldots, \ell_k \in \mathcal{L}$, and $p_i, p_{i+1} \in \ell_i$ for all i (indices taken mod k). Let $(1, t_i, \ldots, t_i^{k-1})$ denote the direction of ℓ_i.



Then

$$p_{i+1} - p_i = a_i(1, t_i, \ldots, t_i^{k-1})$$

for some $a_i \in \mathbb{F}_q \setminus \{0\}$. Thus (recall that $p_{k+1} = p_1$)

$$\sum_{i=1}^{k} a_i(1, t_i, \ldots, t_i^{k-1}) = \sum_{i=1}^{k}(p_{i+1} - p_i) = 0. \tag{1.3}$$

The vectors $(1, t_i, \ldots, t_i^{k-1})$, $i = 1, \ldots, k$, after deleting duplicates, are linearly independent. One way to see this is via the Vandermonde determinant

$$\begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{k-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_k & x_k^2 & \cdots & x_k^{k-1} \end{vmatrix} = \prod_{i<j}(x_j - x_i).$$

For (1.3) to hold, each vector $(1, t_i, \ldots, t_i^{k-1})$ must appear at least twice in the sum, with their coefficients a_i adding up to zero.

Since the lines $\ell_1, \ldots, \ell_k$ are distinct, for each $i = 1, \ldots, k$ (indices taken mod k), the lines ℓ_i and ℓ_{i+1} cannot be parallel. So, $t_i \neq t_{i+1}$. When $k \in \{2, 3, 5\}$ it is impossible to select $t_1, \ldots, t_k$ with no equal consecutive terms (including wraparound) and so that each value is repeated at least twice. Therefore the $2k$-cycle cannot exist. (Why does the argument fail for C_8-freeness?) □

1.11 Randomized Algebraic Constructions

In this section, we show how to add randomness to algebraic constructions, thereby combining the power of both approaches. This idea is due to Bukh (2015).

The algebraic constructions in the previous section can be abstractly described as follows. Take a graph whose vertices are points in some algebraic set (e.g., some finite field geometry), with two vertices x and y being adjacent if some algebraic relationship such as $f(x, y) = 0$ is satisfied. Previously, this f was carefully chosen by hand. The new idea is to take f to be a random polynomial.

We illustrate this technique by giving another proof of the tightness of the KST bound on extremal numbers for $K_{s,t}$ when t is large compared to s.

Theorem 1.11.1 (Tightness of KST bound for large t)
For every $s \geq 2$, there exists some t so that

$$\operatorname{ex}(n, K_{s,t}) \geq \left(\frac{1}{2} - o(1)\right) n^{2-1/s}.$$

The construction we present here has a worse dependence of t on s than in Theorem 1.10.7. The main purpose of this section is to illustrate the technique of randomized algebraic constructions. Bukh (2021) later gave a significant extension of this technique which shows that $\operatorname{ex}(n, K_{s,t}) = \Omega_s(n^{2-1/s})$ for some t close to 9^s, improving on Theorem 1.10.7, which required $t > (s-1)!$.

Proof idea. Take a random polynomial $f(X_1, \ldots, X_s, Y_1, \ldots, Y_s)$ symmetric in the X and Y variables (i.e., $f(X, Y) = f(Y, X)$), but otherwise uniformly chosen among all polynomials with degree up to d with coefficients in $\mathbb{F}_q$. Consider a graph with vertex set $\mathbb{F}_q^s$ and where X and Y are adjacent if $f(X, Y) = 0$.

Given an s-vertex set U, let Z_U denote the set of common neighbors of U. It is an algebraic set: the common zeros of the polynomials $f(X, y)$, $y \in U$. Due to the Lang–Weil bound from algebraic geometry, Z_U is either bounded in size, $|Z_U| \leq C$ (the zero dimensional case), or it must be quite large, say, $|Z_U| > q/2$ (the positive dimensional case). This is unlike an Erdős–Rényi random graph.

One can then deduce, using Markov's inequality, that

$$\mathbb{P}(|Z_U| > C) = \mathbb{P}\left(|Z_U| > \frac{q}{2}\right) \leq \frac{\mathbb{E}[|Z_U|^k]}{(q/2)^k} = \frac{O_k(1)}{(q/2)^k},$$

which is quite small (much smaller compared to an Erdős–Rényi random graph). So typically very few sets U have size $> C$. By deleting these bad Us from the vertex set of the graph, we obtain a $K_{s,C+1}$-free graph with around q^s vertices and on the order of q^{2s-1} edges. ■

Now we begin the actual proof. Let q be the largest prime power satisfying $q^s \leq n$. Due to prime gaps (Theorem 1.10.3), we have $q = (1 - o(1))n^{1/s}$. So it suffices to construct a $K_{s,t}$-free graph on q^s vertices with $(1/2 - o(1))q^{2s-1}$ edges.

Let $d = s^2 + s$ (the reason for this choice will come up later). Let

$$f \in \mathbb{F}_q[X_1, X_2, \ldots, X_s, Y_1, Y_2, \ldots, Y_s]_{\leq d}$$

be a polynomial chosen uniformly at random among all polynomials with degree at most d in each of $X = (X_1, X_2, \ldots, X_s)$ and $Y = (Y_1, Y_2, \ldots, Y_s)$ and furthermore satisfying

$f(X,Y) = f(Y,X)$. In other words,

$$f = \sum_{\substack{i_1+\cdots+i_s \le d \\ j_1+\cdots+j_s \le d}} a_{i_1,\dots,i_s,j_1,\dots,j_s} X_1^{i_1} \cdots X_s^{i_s} Y_1^{j_1} \cdots Y_s^{j_s}$$

where the coefficients $a_{i_1,\dots,i_s,j_1,\dots,j_s} \in \mathbb{F}_q$ are chosen subject to $a_{i_1,\dots,i_s,j_1,\dots,j_s} = a_{j_1,\dots,j_s,i_1,\dots,i_s}$ but otherwise independently and uniformly at random.

Let G be the graph with vertex set $\mathbb{F}_q^s$, with distinct $x, y \in \mathbb{F}_q^s$ adjacent if and only if $f(x,y) = 0$.

Then G is a random graph. The next two lemmas show that G behaves in some ways like a random graph with edges independently appearing with probability $1/q$. Indeed, the next lemma shows that every pair of vertices form an edge with probability $1/q$.

Lemma 1.11.2 (Random polynomial)
Suppose f is randomly chosen as above. For all $u, v \in \mathbb{F}_q^s$,

$$\mathbb{P}[f(u,v) = 0] = \frac{1}{q}.$$

Proof. Note that resampling the constant term of f does not change its distribution. Thus, $f(u,v)$ is uniformly distributed in $\mathbb{F}_q$ for a fixed (u,v). Hence $f(u,v)$ takes each value with probability $1/q$. □

More generally, we show below that the expected occurrence of small subgraphs mirrors that of the usual random graph with independent edges. We write $\binom{U}{2}$ for the set of unordered pairs of element from U.

Lemma 1.11.3 (Random polynomial)
Suppose f is randomly chosen as above. Let $W \subseteq \mathbb{F}_q^s$ with $|W| \le d+1$. Then the vector $(f(u,v))_{\{u,v\} \in \binom{W}{2}}$ is uniformly distributed in $\mathbb{F}_q^{\binom{W}{2}}$. In particular, for any $E \subseteq \binom{W}{2}$, one has

$$\mathbb{P}[f(u,v) = 0 \text{ for all } \{u,v\} \in E] = q^{-|E|}.$$

Proof. We first perform multivariate Lagrange interpolation to show that $(f(u,v))_{\{u,v\}}$ can take all possible values. For each pair $u, v \in W$ with $u \ne v$, we can find some polynomial $\ell_{u,v} \in \mathbb{F}[X_1, \dots, X_s]$ of degree 1 such that $\ell_{u,v}(u) = 1$ and $\ell_{u,v}(v) = 0$. For each $u \in W$, let

$$q_u(X) = \prod_{v \in W \setminus \{u\}} \ell_{u,v}(X) \in \mathbb{F}[X_1, \dots, X_s]$$

which has degree $|W| - 1 \le d$. It satisfies $q_u(u) = 1$, and $q_u(v) = 0$ for all $v \in W \setminus \{u\}$.

Let

$$p(X,Y) = \sum_{\{u,v\} \in \binom{W}{2}} c_{u,v}(q_u(X)q_v(Y) + q_v(X)q_u(Y))$$

with $c_{u,v} \in \mathbb{F}_q$. Note that $p(X,Y) = p(Y,X)$. Also, $p(u,v) = c_{u,v}$ for all distinct $u, v \in W$.

Now let each $c_{u,v} \in \mathbb{F}_q$ above be chosen independently and uniformly at random so that $p(X,Y)$ is a random polynomial. Note that $f(X,Y)$ and $p(X,Y)$ are independent random polynomials both with degree at most d in each of X and Y. Since f is chosen uniformly at random, it has the same distribution as $f + p$. Since $(p(u, v))_{u,v} = (c_{u,v})_{u,v} \in \mathbb{F}_q^{\binom{|W|}{2}}$ is uniformly distributed, the same must be true for $(f(u, v))_{u,v}$ as well. □

Now fix $U \subseteq \mathbb{F}_q^s$ with $|U| = s$. We want to show that it is rare for U to have many common neighbors. We will use the method of moments. Let

$$\begin{aligned} Z_U &= \text{the set of common neighbors of } U \\ &= \{x \in \mathbb{F}_q^s \setminus U : f(x, u) = 0 \text{ for all } u \in U\}. \end{aligned}$$

Then using Lemma 1.11.3, for any $k \le s^2 + 1$,

$$\begin{aligned} \mathbb{E}[|Z_U|^k] &= \mathbb{E}\Big[\Big(\sum_{v \in \mathbb{F}_q^s \setminus U} 1\{v \in Z_U\}\Big)^k\Big] \\ &= \sum_{v^{(1)}, \dots, v^{(k)} \in \mathbb{F}_q^s \setminus U} \mathbb{E}[1\{v^{(1)}, \dots, v^{(k)} \in Z_U\}] \\ &= \sum_{v^{(1)}, \dots, v^{(k)} \in \mathbb{F}_q^s \setminus U} \mathbb{P}[f(u, v) = 0 \text{ for all } u \in U \text{ and } v \in \{v^{(1)}, \dots, v^{(k)}\}] \\ &= \sum_{v^{(1)}, \dots, v^{(k)} \in \mathbb{F}_q^s \setminus U} q^{-|U|\,\#\{v^{(1)}, \dots, v^{(k)}\}}, \end{aligned}$$

with the final step due to Lemma 1.11.3 applied with $W = U \cup \{v^{(1)}, \dots, v^{(k)}\}$, which has cardinality $\le |U| + k \le s + s^2 + 1 = d + 1$. Note that $\#\{v^{(1)}, \dots, v^{(k)}\}$ counts distinct elements in the set. Thus, continuing the above calculation,

$$\begin{aligned} &= \sum_{r \le k} \binom{q^s - |U|}{r} q^{-rs} \#\{\text{surjections } [k] \to [r]\} \\ &\le \sum_{r \le k} \#\{\text{surjections } [k] \to [r]\} \\ &= O_k(1). \end{aligned}$$

Applying the preceding with $k = s^2 + 1$ and using Markov's inequality, we get

$$\mathbb{P}(|Z_U| \ge \lambda) = \mathbb{P}(|Z_U|^{s^2+1} \ge \lambda^{s^2+1}) \le \frac{\mathbb{E}\big[|Z_U|^{s^2+1}\big]}{\lambda^{s^2+1}} \le \frac{O_s(1)}{\lambda^{s^2+1}}. \tag{1.4}$$

***Remark* 1.11.4.** All the probabilistic arguments up to this point would be identical had we used a random graph with independent edges appearing with probability p. In both settings, $|Z_U|$ is a random variable with constant order expectation. However, their distributions are extremely different, as we will soon see. For a random graph with independent edges, $|Z_U|$ behaves like a Poisson random variable, and consequently, for any constant t, $\mathbb{P}(|Z_U| \ge t)$ is bounded from below by a constant. Consequently, many s-element sets of vertices are expected to have at least t common neighbors, and so this method will not work. However, this is not the case with the random algebraic construction. It is impossible for $|Z_U|$ to take on certain ranges of values. If $|Z_U|$ is somewhat large, then it must be very large.

Note that Z_U is defined by s polynomial equations. The next result tells us that the number of points on such an algebraic variety must be either bounded or at least around q.

Lemma 1.11.5 (Dichotomy: number of common zeros)
For all s, d there exists a constant C such that if $f_1(X), \dots, f_s(X)$ are polynomials on $\mathbb{F}_q^s$ of degree at most d, then

$$\{x \in \mathbb{F}_q^s : f_1(x) = \dots f_s(x) = 0\}$$

has size either at most C or at least $q - C\sqrt{q}$.

The lemma can be deduced from the following important result from algebraic geometry due to Lang and Weil (1954), which says that the number of points of an r-dimensional algebraic variety in $\mathbb{F}_q^s$ is roughly q^r, as long as certain irreducibility hypotheses are satisfied. We include here the statement of the Lang–Weil bound. Here $\overline{\mathbb{F}}_q$ denotes the algebraic closure of $\mathbb{F}_q$.

Theorem 1.11.6 (Lang–Weil bound)
Let $g_1, \dots, g_m \in \mathbb{F}_q[X]$ be polynomials of degree at most d. Let

$$V = \left\{x \in \overline{\mathbb{F}}_q^s : g_1(x) = g_2(x) = \dots = g_m(x)\right\}.$$

Suppose V is an irreducible variety. Then

$$\left|V \cap \mathbb{F}_q^s\right| = q^{\dim V}(1 + O_{s,m,d}(q^{-1/2})).$$

The two cases in Lemma 1.11.5 then correspond to the zero-dimensional case and the positive-dimensional case, though some care is needed to deal with what happens if the variety is reducible in the field closure. We refer the reader to Bukh (2015) for details on how to deduce Lemma 1.11.5 from the Lang–Weil bound.

Now, continuing our proof of Theorem 1.11.1. Recall $Z_U = \{x \in \mathbb{F}_q^s \setminus U : f(x,u) = 0 \text{ for all } u \in U\}$. Apply Lemma 1.11.5 to the polynomials $f(X,u)$, $u \in U$. Then for large enough q there exists a constant C from Lemma 1.11.5 such that either $|Z_U| \le C$ (bounded) or $|Z_U| > q/2$ (very large). Thus, by (1.4),

$$\mathbb{P}(|Z_U| > C) = \mathbb{P}\left(|Z_U| > \frac{q}{2}\right) \le \frac{O_s(1)}{(q/2)^{s^2+1}}.$$

So the expected number of s-element subset U with $|Z_U| > C$ is

$$\le \binom{q^s}{s}\frac{O_s(1)}{(q/2)^{s^2+1}} = O_s(1/q).$$

Remove from G a vertex from every s-element U with $|Z_U| > C$. Then the resulting graph is $K_{s,\lceil C\rceil+1}$-free. Since we remove at most q^s edges for each deleted vertex, the expected number of remaining edges is at least

$$\frac{1}{q}\binom{q^s}{2} - O_s(q^{s-1}) = \left(\frac{1}{2} - o(1)\right)q^{2s-1}.$$

Finally, given n, we can take the largest prime q satisfying $q^s \le n$ to finish the proof of Theorem 1.11.1.

Further Reading

Graph theory is a huge subject. There are many important topics that are quite far from the main theme of this book. For a standard introduction to the subject (especially on more classical aspects), several excellent graph theory textbooks are available: Bollobás (1998), Bondy and Murty (2008), Diestel (2017), West (1996). The three-volume *Combinatorial Optimization* by Schrijver (2003) is also an excellent reference for graph theory, with a focus on combinatorial algorithms.

The following surveys discuss in more depth various topics encountered in this chapter:

- *The History of Degenerate (Bipartite) Extremal Graph problems* by Füredi and Simonovits (2013);
- *Hypergraph Turán Problems* by Keevash (2011);
- *Dependent Random Choice* by Fox and Sudakov (2011).

Chapter Summary

- **Turán number** $\mathrm{ex}(n, H)$ = the maximum number of edges in an n-vertex H-free graph.
- **Turán's theorem.** Among all n-vertex K_{r+1}-free graphs, the Turán graph $T_{n,r}$ (a complete r-partite graph with nearly equal sized parts) uniquely maximizes the number of edges.
- **Erdős–Stone–Simonovits Theorem.** For any fixed graph H,

$$\mathrm{ex}(n, H) = \left(1 - \frac{1}{\chi(H) - 1} + o(1)\right) \frac{n^2}{2}.$$

- **Supersaturation** (from one copy to many copies): an n-vertex graph with $\geq \mathrm{ex}(n, H) + \varepsilon n^2$ edges has $\geq \delta n^{v(H)}$ copies of H, for some constant $\delta > 0$ only depending on $\varepsilon > 0$, and provided that n is sufficiently large.
- **Kővári–Sós–Turán theorem.** For fixed $s \leq t$,

$$\mathrm{ex}(n, K_{s,t}) = O_{s,t}(n^{2-1/s}).$$

  - Tight for $K_{2,2}$, $K_{3,3}$, and more generally, for $K_{s,t}$ with t much larger than s (algebraic constructions).
  - Conjectured to be tight in general.
- **Even cycles.** For any integer $k \geq 2$,

$$\mathrm{ex}(n, C_{2k}) = O_k(n^{1+1/k}).$$

  - Tight for $k \in \{2, 3, 5\}$ (algebraic constructions).
  - Conjectured to be tight in general.
- **Randomized constructions** for constructing H-free graphs: destroying all copies of H from a random graph.
- **Algebraic construction**: define edges using polynomials over $\mathbb{F}_q^n$.
- **Randomized algebraic constructions**: randomly select the polynomials.