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Appetizer: Triangles and Equations

Chapter Highlights

- Schur's theorem on monochromatic solutions to $x + y = z$ and its graph theoretic proof
- Problems and results on progressions (e.g., Szemerédi's theorem, the Green–Tao theorem)
- Introduction to the connection between graph theory and additive combinatorics

0.1 Schur's Theorem

Can we prove Fermat's Last Theorem by reducing the equation $X^n + Y^n = Z^n$ modulo a prime p?

It turns out this approach can never work. Dickson (1909) showed that, for every n, the equation mod p can always be solved for sufficiently large primes p. Schur (1916) gave a simpler proof of this result by proving the following theorem, showing that Dickson's result is much more about combinatorics than about number theory.

Theorem 0.1.1 (Schur's theorem)
If the positive integers are colored using finitely many colors, then there is always a monochromatic solution to $x + y = z$ (i.e., x, y, z all have the same color).

We will prove Schur's theorem shortly.

Finitary Versus Infinitary

Many theorems in this book can be stated in multiple equivalent ways. For instance, Schur's theorem was stated in Theorem 0.1.1 an **infinitary** form. Following is an equivalent **finitary** version. We write $[N] := \{1, 2, \ldots, N\}$.

Theorem 0.1.2 (Schur's theorem, finitary version)
For every positive integer r, there exists a positive integer $N = N(r)$ such that if each element of $[N]$ is colored using one of r colors, then there is a monochromatic solution to $x + y = z$.

The finitary formulation leads to quantitative questions. For example, how large does $N(r)$ have to be as a function of r? Questions of this type are often quite difficult to resolve, even approximately. There are lots of open questions concerning quantitative bounds.

Proof that the preceding two formulations of Schur's theorem are equivalent. First, the finitary version (Theorem 0.1.2) of Schur's theorem easily implies the infinitary version (Theorem 0.1.1). Indeed, in the infinitary version, given a coloring of the positive integers, we can consider the colorings of the first $N(r)$ integers and use the finitary statement to find a monochromatic solution.

To prove that the infinitary version implies the finitary version, we use a **diagonalization argument**. Fix r, and suppose that for every N there is some coloring $\phi_N \colon [N] \to [r]$ that avoids monochromatic solutions to $x + y = z$. We can take an infinite subsequence of (ϕ_N) such that, for every $k \in \mathbb{N}$, the value of $\phi_N(k)$ stabilizes to a constant as N increases along this subsequence. (We can do this by repeatedly restricting to convergent infinite subsequences.) Then the ϕ_Ns, along this subsequence, converge pointwise to some coloring $\phi \colon \mathbb{N} \to [r]$, avoiding monochromatic solutions to $x+y=z$, but ϕ contradicts the infinitary statement. □

Fermat's Equation Modulo a Prime

Let us show how to deduce the existence of solutions to $X^n + Y^n \equiv Z^n \pmod p$ using Schur's theorem.

Theorem 0.1.3 (Fermat's Last Theorem mod p)

Let n be a positive integer. For all sufficiently large prime p, there exist $X, Y, Z \in \{1, \ldots, p-1\}$ such that $X^n + Y^n \equiv Z^n \pmod p$.

Proof assuming Schur's theorem (Theorem 0.1.2). Let $(\mathbb{Z}/p\mathbb{Z})^\times$ denote the group of nonzero residues mod p under multiplication. Let $H = \{x^n : x \in (\mathbb{Z}/p\mathbb{Z})^\times\}$ be the subgroup of nth powers in $(\mathbb{Z}/p\mathbb{Z})^\times$. Since $(\mathbb{Z}/p\mathbb{Z})^\times$ is a cyclic group of order $p-1$ (due to the existence of primitive roots mod p, a fact from elementary number theory), the index of H in $(\mathbb{Z}/p\mathbb{Z})^\times$ is equal to $\gcd(n, p-1) \le n$. So the cosets of H partition $\{1, 2, \ldots, p-1\}$ into $\le n$ sets. Viewing each of the $\le n$ cosets of H as a "color," by the finitary statement of Schur's theorem (Theorem 0.1.2), for p large enough as a function of n, there exists a solution to

$$x + y = z \quad \text{in } \mathbb{Z}$$

in some coset of H, say $x, y, z \in aH$ for some $a \in (\mathbb{Z}/p\mathbb{Z})^\times$. Since H consists of nth powers, we have $x = aX^n$, $y = aY^n$, and $z = aZ^n$ for some $X, Y, Z \in (\mathbb{Z}/p\mathbb{Z})^\times$. Thus

$$aX^n + aY^n \equiv aZ^n \pmod p.$$

Since $a \in (\mathbb{Z}/p\mathbb{Z})^\times$ is invertible mod p, we have $X^n + Y^n \equiv Z^n \pmod p$ as desired. □

Ramsey's Theorem

Now let us prove Schur's theorem (Theorem 0.1.2) by deducing it from an analogous result about edge-coloring of a complete graph. We write K_N for the complete graph on N vertices.

Theorem 0.1.4 (Multicolor triangle Ramsey theorem)

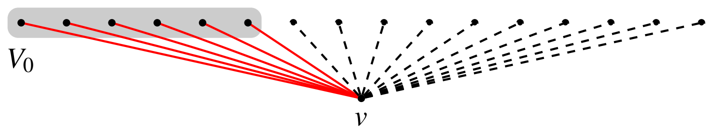
For every positive integer r, there is some integer $N = N(r)$ such that if each edge of K_N is colored using one of r colors, then there is a monochromatic triangle.

Proof. Define

$$N_1 = 3, \quad \text{and} \quad N_r = r(N_{r-1} - 1) + 2 \text{ for all } r \geq 2. \tag{0.1}$$

We show by induction on r that every coloring of the edges of K_{N_r} by r colors has a monochromatic triangle. The case $r = 1$ holds trivially.

Suppose the claim is true for $r - 1$ colors. Consider any edges-coloring of K_{N_r} using r colors. Pick an arbitrary vertex v. Of the $N_r - 1 = r(N_{r-1} - 1) + 1$ edges incident to v, by the pigeonhole principle, at least N_{r-1} edges incident to v have the same color, say red. Let V_0 be the vertices joined to v by a red edge.



If there is a red edge inside V_0, we obtain a red triangle. Otherwise, there are at most $r - 1$ colors appearing among $|V_0| \geq N_{r-1}$ vertices, and we have a monochromatic triangle inside V_0 by the induction hypothesis. □

Exercise 0.1.5. Show that N_r from (0.1) satisfies $N_r = 1 + r! \sum_{i=0}^{r} 1/i! = \lceil r!e \rceil$.

***Remark* 0.1.6** (Ramsey's theorem)**.** The preceding recursive/inductive pigeonhole argument can be easily adapted to prove Ramsey's theorem in general.

Theorem 0.1.7 (Graph Ramsey theorem)

For every k and r there exists some $N = N(k, r)$ such that if each edge of K_N is colored using one of r colors, then there is a monochromatic K_k.

Exercise 0.1.8. Prove the graph Ramsey theorem (Theorem 0.1.7).

Ramsey's theorem extends even more generally to hypergraphs.

Theorem 0.1.9 (Hypergraph Ramsey theorem)

For every k, r, s there exists some $N = N(k, r, s)$ such that if each edge of a complete s-uniform hypergraph on N vertices is colored using one of r colors, then there is a monochromatic clique on k vertices.

Exercise 0.1.10. Prove the hypergraph Ramsey theorem (Theorem 0.1.9).

Remark* 0.1.11** (Bounds for multicolor triangle Ramsey numbers)**.** The smallest $N(r)$ in Theorem 0.1.4 is also known as the ***multicolor triangle Ramsey number, denoted $\boldsymbol{R(3, 3, \ldots, 3)}$ with 3 repeated r times. It is a major open problem in Ramsey theory to determine the rate of growth of this Ramsey number. Here is an easy argument showing an exponential lower bound. (Compare it to the upper bound from Exercise 0.1.5.)

Proposition 0.1.12 (Multicolor triangle Ramsey numbers: exponential lower bound)

For each positive integer r, there exists an edge-coloring of K_{2^r} using r colors with no monochromatic triangle.

Proof. Label the vertices by elements of $\{0,1\}^r$. Assign an edge color i if i is the smallest index such that the two endpoint vertices differ on coordinate i. This coloring does not have monochromatic triangles. Indeed, suppose x, y, z form a monochromatic triangle with color i, then $x_i, y_i, z_i \in \{0,1\}$ must be all distinct, which is impossible. $\square$

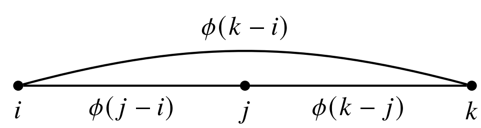
Schur (1916) had actually given an even better lower bound: see Exercise 0.1.14. One of Erdős' favorite problems asks whether there is an exponential upper bound. This is a major open problem in Ramsey theory, and it is related to to other important topics in combinatorics such as the Shannon capacity of graphs (see, e.g., the survey by Nešetřil and Rosenfeld 2001).

Open Problem 0.1.13 (Multicolor triangle Ramsey numbers: exponential upper bound) Is there a constant $C > 0$ so that if $N \geq C^r$, then every edge-coloring of K_N using r colors contains a monochromatic triangle?

Graph Theoretic Proof of Schur's Theorem

We set up a graph whose triangles correspond to solutions to $x + y = z$, and then apply the multicolor triangle Ramsey theorem.

Proof of Schur's theorem (Theorem 0.1.2). Let $\phi\colon [N] \to [r]$ be a coloring. Color the edges of a complete graph with vertices $\{1, \ldots, N+1\}$ by giving the edge $\{i, j\}$ with $i < j$ the color $\phi(j - i)$.



By Theorem 0.1.4, if N is large enough, then there is a monochromatic triangle, say on vertices $i < j < k$. So, $\phi(j - i) = \phi(k - j) = \phi(k - i)$. Take $x = j - i$, $y = k - j$, and $z = k - i$. Then $\phi(x) = \phi(y) = \phi(z)$ and $x + y = z$, as desired. $\square$

Now that we proved Schur's theorem, let us pause and think about what we gained by translating the problem to graph theory. We were able to apply Ramsey's theorem, whose proof considers restrictions to subgraphs, which would have been rather unnatural if we had worked exclusively in the integers. Graphs gave us greater flexibility.

Later in the book, we will see other more sophisticated examples of this idea. We will gain new perspectives by bringing number theory problems to graph theory.

Exercise 0.1.14 (Schur's lower bound)**.** Let $N(r)$ denote the smallest positive integer in Schur's theorem (Theorem 0.1.2). Show that $N(r) \geq 3N(r-1) - 1$ for every r. Deduce that $N(r) \geq (3^r + 1)/2$ for every r. Also deduce that there exists a coloring of the edges of $K_{(3^r+1)/2}$ with r colors so that there are no monochromatic triangles.

Exercise 0.1.15 (Upper bound on Ramsey numbers)**.** Let s and t be positive integers. Show that if the edges of a complete graph on $\binom{s+t-2}{s-1}$ vertices are colored with red and blue, then there must be either a red K_s or a blue K_t.

Exercise 0.1.16 (Monochromatic triangles compared to random coloring)**.**

(a) True or false: If the edges of K_n are colored using 2 colors, then at least $1/4 - o(1)$ fraction of all triangles are monochromatic. (Note that $1/4$ is the fraction one expects if the edges were colored uniformly at random.)

(b) True or false: if the edges of K_n are colored using 3 colors, then at least $1/9 - o(1)$ fraction of all triangles are monochromatic.

(c*) True or false: if the edges of K_n are colored using 2 colors, then at least $1/32 - o(1)$ fraction of all copies of K_4s are monochromatic.

0.2 Progressions

Additive combinatorics describes a rapidly growing body of mathematics motivated by simple-to-state questions about addition and multiplication of integers. (The name "additive combinatorics" became popular in the 2000's, when the field witnessed a rapid explosion thanks to the groundbreaking works of Gowers, Green, Tao, and others; previously the area was more commonly known as "combinatorial number theory.") The problems and methods in additive combinatorics are deep and far-reaching, connecting many different areas of mathematics such as graph theory, harmonic analysis, ergodic theory, discrete geometry, and model theory.

Here we highlight some important developments in additive combinatorics, particularly concerning progressions. The ideas behind these developments form some of the core themes of this book.

Towards Szemerédi's Theorem

Schur's theorem is one of the earliest results in additive combinatorics. It has important variations and extensions, such as the following seminal result of van der Waerden (1927) on monochromatic arithmetic progressions.

Theorem 0.2.1 (van der Waerden's theorem)

If the integers are colored using finitely many colors, then there exist arbitrarily long monochromatic arithmetic progressions.

Note that having arbitrarily long arithmetic progressions is very different from having infinitely long arithmetic progressions, as seen in the next exercise.

Exercise 0.2.2. Show that $\mathbb{Z}$ may be colored using two colors so that it contains no infinitely long arithmetic progressions.

Erdős and Turán (1936) conjectured a stronger statement, that any subset of the integers with positive density contains arbitrarily long arithmetic progressions. To be precise, we say that $A \subseteq \mathbb{Z}$ has ***positive upper density*** if

$$\limsup_{N\to\infty} \frac{|A \cap \{-N, \dots, N\}|}{2N+1} > 0. \tag{0.2}$$

(There are several variations of definition of density – the exact formulation is not crucial here.) The Erdős and Turán conjecture speculates that the "true" reason for van der Waerden's theorem is not so much having finitely many colors (as in Ramsey's theorem), but rather that some color class necessarily has positive density. (The analogous claim is false for graphs since a triangle-free graph can have edge-density up to $1/2$; we explore this topic further in the next chapter.)

Roth (1953) proved the Erdős and Turán conjecture for 3-term arithmetic progressions using Fourier analysis. It took another two decades before Szemerédi (1975) fully settled the conjecture in a combinatorial tour de force. These theorems by Roth and Szemerédi are landmark results in additive combinatorics. Much of what we will discuss in the book is motivated by these results and the developments around them.

Theorem 0.2.3 (Roth's theorem)
Every subset of the integers with positive upper density contains a 3-term arithmetic progression.

Theorem 0.2.4 (Szemerédi's theorem)
Every subset of the integers with positive upper density contains arbitrarily long arithmetic progressions.

Szemerédi's theorem is deep and intricate. This important work led to many subsequent developments in additive combinatorics. Several different proofs of Szemerédi's theorem have since been discovered, and some of them have blossomed into rich areas of mathematical research. Here are some of the most influential modern proofs of Szemerédi's theorem (in historical order):

- The ergodic theoretic approach by Furstenberg (1977);
- Higher-order Fourier analysis by Gowers (2001);
- Hypergraph regularity independently by Rödl et al. (2005) and Gowers (2001).

Another modern proof of Szemerédi's theorem results from the **density Hales–Jewett theorem**, which was originally proved by Furstenberg and Katznelson (1978) using ergodic theory. Subsequently a new combinatorial proof was found in the first successful Polymath Project (Polymath 2012), an online collaborative project initiated by Gowers.

Each approaches has its own advantages and disadvantages. For example, the ergodic approach led to multidimensional and polynomial generalizations of Szemerédi's theorem, which we discuss in what follows. On the other hand, the ergodic approach does not give any concrete quantitative bounds. Fourier analysis and its generalizations produce the best quantitative bounds to Szemerédi's theorem. They also led to deep results about counting patterns in the prime numbers. However, there appear to be difficulties and obstructions extending Fourier analysis to higher dimensions.

The relationships between these different approaches to Szemerédi's theorem are not yet completely understood. A unifying theme underlying all known approaches to Szemerédi's theorem is

the dichotomy between structure and pseudorandomness.

This phrase is popularized by Tao (2007b) and others. It will be a theme throughout this book. We will see facets of this dichotomy in both graph theory and additive combinatorics.

Quantitative Bounds on Szemerédi's Theorem

There is much interest in obtaining better quantitative bounds on Szemerédi's theorem. Roth's initial proof showed that every subset of $[N]$ avoiding 3-term arithmetic progressions has size $O(N/\log\log N)$. (We will see this proof in Chapter 6.) Roth's upper bound has been improved steadily over time, all via refinement of his Fourier analytic technique. At the time of this writing, the current best upper bound is $N/(\log N)^{1+c}$ for some constant $c > 0$ (Bloom and Sisask 2020). For 4-term arithmetic progressions, the best known upper bound is $N/(\log N)^c$ (Green and Tao 2017). For k-term arithmetic progressions, with fixed $k \geq 5$, the best known upper bound is $N/(\log\log N)^{c_k}$ (Gowers 2001).

As for lower bounds, Behrend (1946) constructed a subset of $[N]$ of size $Ne^{-c\sqrt{\log N}}$ that avoids 3-term arithmetic progressions. This is an important construction that we will see in Section 2.5. Some researchers think that this lower bound is closer to the truth, since for a variant of Roth's theorem (namely avoiding solutions to $x + y + z = 3w$), Behrend's construction is quite close to the truth (Schoen and Shkredov 2014; Schoen and Sisask 2016).

Erdős famously conjectured the following.

Conjecture 0.2.5 (Erdős conjecture on arithmetic progressions)
Every subset A of integers with $\sum_{a\in A} 1/a = \infty$ contains arbitrarily long arithmetic progressions.

This is a strengthening of the Erdős–Turán conjecture (later Szemerédi's theorem), since every subset of integers with positive density necessarily has a divergent harmonic sum. Erdős' conjecture was motivated by the primes (see the Green–Tao theorem in what follows). It has an attractive statement and is widely publicized. The supposed connection between divergent harmonic series and arithmetic progressions seems magical. However, this connection is perhaps somewhat misleading. The hypothesis on divergent harmonic series implies that there are infinitely many N for which $|A \cap [N]| \geq N/(\log N(\log\log N)^2)$. So the Erdős conjecture is really about an upper bound on Szemerédi's theorem. As mentioned earlier, it is plausible that the true upper bound for Szemerédi may be much lower than $1/\log N$. Nevertheless, the "logarithmic barrier" proposed by the Erdős conjecture has a special symbolic and historical status. Erdős' conjecture for k-term arithmetic progressions is now proved for $k = 3$ thanks to the new $N/(\log N)^{1+c}$ upper bound (Bloom and Sisask 2020), but it remains very much open for all $k \geq 4$.

Improving quantitative bounds on Szemerédi's theorem remains an active area of research. Perhaps by the time you read this book (or when I update it to a future edition), these bounds will have been significantly improved.

Extensions of Szemerédi's Theorem

Instead of working over subsets of integers, what happens if we consider subsets of the lattice $\mathbb{Z}^d$? We say that $A \subseteq \mathbb{Z}^d$ has ***positive upper density*** if

$$\limsup_{N\to\infty} \frac{|A \cap [-N, N]^d|}{(2N+1)^d} > 0$$

(as before, other similar definitions are possible). How can we generalize the notion of a subset of $\mathbb{Z}$ containing arbitrarily long arithmetic progressions? We could desire A to contain $k \times k \times \cdots \times k$ cubical grids for arbitrarily large k. Equivalently, we say that $A \subseteq \mathbb{Z}^d$ ***contains arbitrary constellations*** if for every finite set $F \subseteq \mathbb{Z}^d$, there is some $a \in \mathbb{Z}^d$ and $t \in \mathbb{Z}_{>0}$ such that $a + t \cdot F = \{a + tx : x \in F\}$ is contained in A. In other words, A contains every finite pattern F (allowing dilation and translation, as captured by $a + t \cdot F$). The following multidimensional generalization of Szemerédi's theorem was proved by Furstenberg and Katznelson (1978) using ergodic theory, though a combinatorial proof was later discovered as a consequence of the hypergraph regularity method.

Theorem 0.2.6 (Multidimensional Szemerédi theorem)
Every subset of $\mathbb{Z}^d$ of positive upper density contains arbitrary constellations.

For example, the theorem implies that every subset of $\mathbb{Z}^2$ of positive upper density contains a $k \times k$ axis-aligned square grid for every k.

There is also a polynomial extension of Szemerédi's theorem. Let us first state a special case, originally conjectured by Lovász and proved independently by Furstenberg (1977) and Sárkőzy (1978).

Theorem 0.2.7 (Furstenberg–Sárkőzy theorem)
Any subset of the integers with positive upper density contains two numbers differing by a perfect square.

In other words, the set always contains $\{x, x + y^2\}$ for some $x \in \mathbb{Z}$ and $y \in \mathbb{Z}_{>0}$. What about other polynomial patterns? The following polynomial generalization was proved by Bergelson and Leibman (1996).

Theorem 0.2.8 (Polynomial Szemerédi theorem)
Suppose $A \subseteq \mathbb{Z}$ has positive upper density. If $P_1, \ldots, P_k \in \mathbb{Z}[X]$ are polynomials with $P_1(0) = \cdots = P_k(0) = 0$, then there exist $x \in \mathbb{Z}$ and $y \in \mathbb{Z}_{>0}$ such that $x + P_1(y), \ldots, x + P_k(y) \in A$.

In fact, Bergelson and Leibman proved a common generalization – a multidimensional polynomial Szemerédi theorem. (Can you guess what it says?)

We will not discuss the polynomial Szemerédi theorem in this book. Currently the only known proof of the most general form of the polynomial Szemerédi theorem uses ergodic theory, though quantitative bounds are known for certain patterns (e.g., Peluse (2020)).

Building on Szemerédi's theorem as well as other important developments in number theory, Green and Tao (2008) proved their famous theorem that settled an old folklore conjecture about prime numbers. Their theorem is one of the most celebrated mathematical achievements of this century.

Theorem 0.2.9 (Green–Tao theorem)
The primes contain arbitrarily long arithmetic progressions.

We will discuss the Green–Tao theorem in Chapter 9. The theorem has been extended to polynomial progressions (Tao and Ziegler 2008) and to higher dimensions (Tao and Ziegler 2015; also see Fox and Zhao 2015).

0.3 What's Next in the Book?

One of our goals is to understand two different proofs of Roth's theorem, which has the following finitary statement. We say that a set is ***3-AP-free*** if it does not contain a 3-term arithmetic progression.

Theorem 0.3.1 (Roth's theorem)
Every 3-AP-free subset of $[N]$ has size $o(N)$.

Roth originally proved his result using **Fourier analysis** (also called the **Hardy–Littlewood circle method** in this context). We will see Roth's proof in Chapter 6.

In the 1970's, Szemerédi developed the **graph regularity method**. It is now a central technique in extremal graph theory. Ruzsa and Szemerédi (1978) used the graph regularity method to give a new graph theoretic proof of Roth's theorem. We will see this proof as well as other applications of the graph regularity method in Chapter 2.

Extremal graph theory, broadly speaking, concerns questions of the form: what is the maximum (or minimum) possible number of some structure in a graph with certain prescribed properties? A starting point (historically and also pedagogically) in extremal graph theory is the following question:

Question 0.3.2 (Triangle-free graphs)
What is the maximum number of edges in a triangle-free n-vertex graph?

This question has a relatively simple answer, and it will be the first topic in the next chapter. We will then explore related questions about the maximum number of edges in a graph without some given subgraph.

Although Question 0.3.2 above sounds similar to Roth's theorem, it does not actually allow us to deduce Roth's theorem. Instead, we need to consider the following question.

Question 0.3.3
What is the maximum number of edges in an n-vertex graph where every edge is contained in a unique triangle?

This innocent looking question turns out to be incredibly mysterious. In Chapter 2, we develop the graph regularity method and use it to prove that any such graph must have $o(n^2)$ edges. And we then deduce Roth's theorem from this graph theoretic claim.

The graph regularity method illustrates the dichotomy of structure and pseudorandomness in graph theory. Some of the later chapters dive further into related concepts. Chapter 3 explores **pseudorandom graphs** – what does it mean for a graph to look random? Chapter 4 concerns **graph limits**, a convenient analytic language for capturing many important concepts in earlier chapters. Chapter 5 explores **graph homomorphism inequalities**, revisiting questions from extremal graph theory with an analytic lens.

And then we switch gears (but not entirely) to some core topics in additive combinatorics.

Chapter 6 contains the Fourier analytic proof of **Roth's theorem**. There will be many thematic similarities between elements of the Fourier analytic proof and earlier topics. Chapter 7 explores the structure of set addition. Here we prove **Freiman's theorem** on sets with small additive doubling, a cornerstone result in additive combinatorics. It also plays a key role in Gowers' proof of Szemerédi's theorem, generalizing Fourier analysis to higher-order Fourier analysis, although we will not go into the latter topic in this book (see Further Reading at the end of Chapter 7). In Chapter 8, we explore the **sum-product problem**, which is closely connected to incidence geometry (and we will see another graph theoretic proof there). In Chapter 9, we discuss the **Green–Tao theorem** and prove an extension of Szemerédi's theorem to sparse pseudorandom sets, which plays a central role in the proof of the Green–Tao theorem.

I hope that you will enjoy this book. I have been studying this subject since I began graduate school. I still think about these topics nearly every day. My goal is to organize and distill the beautiful mathematics in this field as a friendly introduction.

The chapters do have some logical dependencies, but not many. Each topic can be studied and enjoyed on its own, though, you will gain a lot more by appreciating the overall themes and connections.

There is still a lot that we do not know. Perhaps you too will be intrigued by the boundless open questions that are still waiting to be explored.

Further Reading

The book *Ramsey Theory* by Graham, Rothschild, and Spencer (1990) is a wonderful introduction to the subject. It has beautiful accounts of theorems of Ramsey, van der Waerden, Hales–Jewett, Schur, Rado, and others, that form the foundation of Ramsey theory.

For a survey of modern developments in additive combinatorics, check out the book review by Green (2009a) of *Additive Combinatorics* by Tao and Vu (2006).

Chapter Summary

- **Schur's theorem.** Every coloring of $\mathbb{N}$ using finitely many colors contains a monochromatic solution to $x + y = z$.
  - Proof: set up a graph whose triangles correspond to solutions to $x + y = z$, and then apply **Ramsey's theorem**.
- **Szemerédi's theorem.** Every subset of $\mathbb{N}$ with positive density contains arbitrarily long arithmetic progressions.
  - A foundational result that led to important developments in **additive combinatorics**.
  - Several different proofs, each illustrating the **dichotomy of structure of pseudorandomness** in a different context.
  - Extensions: multidimensional, polynomial, primes (Green–Tao).