

18.225 PROBLEM SET (FALL 2021)

A. RAMSEY

- A1. *Upper bound on Ramsey numbers.* Let s and t be positive integers. Show that if the edges of a complete graph on $\binom{s+t-2}{s-1}$ vertices are colored with red and blue, then there must be either a red K_s or a blue K_t .
- A2. *Ramsey's theorem.* Show that for every s and r there exists some $N = N(s, r)$ such that every coloring of the edges of K_N using r colors, there exists some monochromatic copy clique on s vertices.

Generalize this statement to hypergraphs.

- ps1 A3. Prove that it is possible to color \mathbb{N} using two colors so that there is no infinitely long monochromatic arithmetic progression.
- ps1 A4. *Many monochromatic triangles*
- (a) True or false: If the edges of K_n are colored using 2 colors, then at least $1/4 - o(1)$ fraction of all triangles are monochromatic. (Note that $1/4$ is the fraction one expects if the edges were colored uniformly at random.)
- (b) True or false: if the edges of K_n are colored using 3 colors, then at least $1/9 - o(1)$ fraction of all triangles are monochromatic.

B. FORBIDDING A SUBGRAPH

- ps1 B1. Show that a graph with n vertices and m edges has at least $\frac{4m}{3n} \left(m - \frac{n^2}{4}\right)$ triangles.
- ps1* B2. Prove that every n -vertex graph with at least $\lfloor n^2/4 \rfloor + 1$ edges contains at least $\lfloor n/2 \rfloor$ triangles.
- ps1* B3. Prove that every n -vertex graph with at least $\lfloor n^2/4 \rfloor + 1$ edges contains some edge in at least $(1/6 - o(1))n$ triangles, and that this constant $1/6$ is best possible.
- ps1 B4. Show that every n -vertex triangle-free graph with minimum degree greater than $2n/5$ is bipartite.
- B5. *K_{r+1} -free graphs close to the Turán bound are nearly r -partite*
- ps1 (a) Let G be an n -vertex triangle-free graph with at least $\lfloor n^2/4 \rfloor - k$ edges. Prove that G can be made bipartite by removing at most k edges.
- ps1* (b) Let G be an n -vertex K_{r+1} -free graph with at least $e(T_{n,r}) - k$ edges, where $T_{n,r}$ is the Turán graph. Prove that G can be made r -partite by removing at most k edges.
- ps1* (c) Let G be an n -vertex graph with $\lfloor n^2/4 \rfloor - k$ edges (here $k \in \mathbb{Z}$) and t triangles. Prove that G can be made bipartite by removing at most $k + 6t/n$ edges, and that this constant 6 is best possible.
- ps1* B6. *Large induced bipartite subgraph.* Prove that for every $\epsilon > 0$, there exist $\delta, C > 0$ so that the following holds. If G is an n -vertex graph with at least $n^2/4$ edges such that every edge of G lies in at most $(1/2 - \epsilon)n$ triangles, and the number of triangles t of G is at most δn^3 , then there is an induced bipartite subgraph containing all but at most Ct/n^2 vertices of G .

B7. Let G be a K_{r+1} -free graph. Prove that there is another graph H on the same vertex set as G such that $\chi(H) \leq r$ and $d_H(x) \geq d_G(x)$ for every vertex x (here $d_H(x)$ is the degree of x in H , and likewise with $d_G(x)$ for G). Give another proof of Turán's theorem from this fact.

ps1

B8. Let X and Y be independent and identically distributed random vectors in \mathbb{R}^d according to some arbitrary probability distribution. Prove that

$$\mathbb{P}(|X + Y| \geq 1) \geq \frac{1}{2}\mathbb{P}(|X| \geq 1)^2.$$

B9. Let S be a set of n points in the plane, with the property that no two points are at distance greater than 1. Show that S has at most $\lfloor n^2/3 \rfloor$ pairs of points at distance greater than $1/\sqrt{2}$. Also, show that the bound $\lfloor n^2/3 \rfloor$ is tight (i.e., cannot be improved).

ps1

B10. *Density Ramsey*. Prove that for every s and r , there exist $c > 0$ and n_0 such that for all $n > n_0$, if the edges of K_n are colored using r colors, then at least c fraction of all copies of K_s are monochromatic.

B11. *Density version of Szemerédi's theorem*. Let $k \geq 3$. Assuming Szemerédi's theorem for k -term arithmetic progressions (i.e., every subset of $[N]$ without a k -term arithmetic progression has size $o(N)$), prove the following density version of the Szemerédi's theorem:

For every $\delta > 0$ there exist c and N_0 (both depending only on k and δ) such that every $A \subset [N]$ with $|A| \geq \delta N$ and $N \geq N_0$, the number of k -term arithmetic progressions in A is at least cN^2 .

ps2

B12. (How *not* to define density in a product set) Let $S \subset \mathbb{Z}^2$. Define

$$d_k(S) = \max_{\substack{A, B \subset \mathbb{Z} \\ |A|=|B|=k}} \frac{|S \cap (A \times B)|}{|A||B|}.$$

Show that $\lim_{k \rightarrow \infty} d_k(S)$ exists and is always either 0 or 1.

(Note: You are only allowed to invoke theorems we proved in class.)

B13. Show that, for every $\epsilon > 0$, there exists $\delta > 0$ such that every graph with n vertices and at least ϵn^2 edges contains a copy of $K_{s,t}$ where $s \geq \delta \log n$ and $t \geq n^{0.99}$.

ps2

B14. *Density version of Kővári–Sós–Turán*. Prove that for every positive integers $s \leq t$, there are constants $C, c > 0$ such that every n -vertex graph with $p \binom{n}{2}$ edges contains at least $cp^{st}n^{s+t}$ copies of $K_{s,t}$, provided that $p \geq Cn^{-1/s}$.

B15. *Erdős–Stone theorem for hypergraphs*. Let H be an r -graph. Show that $\pi(H[s]) = \pi(H)$, where $H[s]$, the s -blow-up of H , is obtained by replacing every vertex of H by s duplicates of itself.

ps2

B16. Let T be a tree with k edges. Show that $\text{ex}(n, T) \leq kn$.

ps2

B17. Find a graph H with $\chi(H) = 3$ and $\text{ex}(n, H) > \frac{1}{4}n^2 + n^{1.99}$ for all sufficiently large n .

The next two problems concern the dependent random choice technique.

ps2

B18. Let $\epsilon > 0$. Show that, for sufficiently large n , every K_4 -free graph with n vertices and at least ϵn^2 edges contains an independent set of size at least $n^{1-\epsilon}$.

ps2*

B19. *Extremal numbers of degenerate graphs*

(a) Prove that there is some absolute constant $c > 0$ so that for every positive integer r , every n -vertex graph with at least $n^{2-c/r}$ edges contains disjoint non-empty vertex subsets A

and B such that every subset of at most r vertices in A has at least n^c common neighbors in B and every subset of at most r vertices in B has at least n^c neighbors in A .

- (b) We say that a graph H is r -degenerate if its vertices can be ordered so that every vertex has at most r neighbors that appear before it in the ordering. Show that for every r -degenerate bipartite graph H there is some constant $C > 0$ so that $\text{ex}(n, H) \leq Cn^{2-c/r}$, where c is the same absolute constant from part (a) (c should not depend on H or r).

C. THE GRAPH REGULARITY METHOD

For simplicity, you are welcome to apply the equitable version of Szemerédi's regularity lemma.

C1. *Unavoidability of irregular pairs.* Let the half-graph H_n be the bipartite graph on $2n$ vertices $\{a_1, \dots, a_n, b_1, \dots, b_n\}$ with edges $\{a_i b_j : i \leq j\}$.

- (a) For every $\epsilon > 0$, explicitly construct an ϵ -regular partition of H_n into $O(1/\epsilon)$ parts.
 (b) Show that there is some $c > 0$ such that for every $\epsilon \in (0, c)$, every integer k and sufficiently large multiple n of k , every partition of the vertices of H_n into k equal-sized parts contains at least ck pairs of parts which are not ϵ -regular.

ps2

C2. Show that there is some absolute constant $C > 0$ such that for every $0 < \epsilon < 1/2$, every graph on n vertices contains an ϵ -regular pair of vertex subsets each with size at least δn , where $\delta = 2^{-\epsilon^{-C}}$.

C3. *Existence of a regular set.* Given a graph G , we say that $X \subset V(G)$ is ϵ -regular if the pair (X, X) is ϵ -regular, i.e., for all $A, B \subset X$ with $|A|, |B| \geq \epsilon|X|$, one has $|d(A, B) - d(X, X)| \leq \epsilon$. This problem asks for two different proofs of the claim: for every $\epsilon > 0$, there exists $\delta > 0$ such that every graph contains an ϵ -regular subset of vertices of size at least δ fraction of the vertex set.

ps2

- (a) Prove the claim using Szemerédi's regularity lemma, showing that one can obtain the ϵ -regular subset by combining a suitable sub-collection of parts from some regularity partition.

ps2*

- (b) Give an alternative proof of the claim showing that one can take $\delta = \exp(-\exp(\epsilon^{-C}))$ for some constant C .

ps2*

C4. *Regularity partition into regular sets.* Prove or disprove: for every $\epsilon > 0$ there exists M so that every graph has an ϵ -regular partition into at most M parts, with every part being ϵ -regular with itself.

ps2*

C5. *Arithmetic triangle removal lemma.* Show that for every $\epsilon > 0$, there exists $\delta > 0$ such that if $A \subset [n]$ has fewer than δn^2 many triples $(x, y, z) \in A^3$ with $x + y = z$, then there is some $B \subset A$ with $|A \setminus B| \leq \epsilon n$ such that B is sum-free, i.e., there do not exist $x, y, z \in B$ with $x + y = z$.

ps3

C6. *The (6,3) theorem.* Let H be an n -vertex 3-uniform hypergraph without a subgraph having 6 vertices and 3 edges. Prove that H has $o(n^2)$ edges.

C7. *Ramsey-Turán.*

ps3

- (a) Show that for every $\epsilon > 0$, there exists $\delta > 0$ such that every n -vertex K_4 -free graph with at least $(\frac{1}{8} + \epsilon)n^2$ edges contains an independent set of size at least δn .

- ps3 (b) Show that for every $\epsilon > 0$, there exists $\delta > 0$ such that every n -vertex K_4 -free graph with at least $(\frac{1}{8} - \delta)n^2$ edges and independence number at most δn can be made bipartite by removing at most ϵn^2 edges.
- C8. Show that for every H and $\epsilon > 0$ there exists $\delta > 0$ such that every graph on n vertices without an induced copy of H contains an induced subgraph on at least δn vertices whose edge density is at most ϵ or at least $1 - \epsilon$.
- ps3 C9. Show that for every Δ there exists a constant C_Δ so that if H is a graph with maximum degree at most Δ , then every 2-edge-coloring of a complete graph on at least $C_\Delta v(H)$ vertices contains a monochromatic copy of H .
- ps3* C10. Using regularity, show that the number of n -vertex triangle-free graphs is $2^{(1/4+o(1))n^2}$.
(Your proof should be easily extendable to show that for every fixed graph H , the number of n -vertex H -free graphs is $2^{(\pi(H)+o(1))\binom{n}{2}}$.)
- ps3* C11. Show that for every graph H there is some graph G such that if the edges of G are colored with two colors, then some induced subgraph of G is a monochromatic copy of H .
- ps3* C12. Show that for every $\alpha > 0$, there exists $\beta > 0$ such that every graph on n vertices with at least αn^2 edges contains a d -regular subgraph for some $d \geq \beta n$ (here d -regular refers to every vertex having degree d).
- ps3 C13. *Multidimensional Szemerédi theorem for axis-aligned squares.* Assuming the tetrahedron removal lemma for 3-uniform hypergraphs, deduce that if $A \subset [N]^2$ contains no axes-aligned squares (i.e., four points of the form $(x, y), (x + d, y), (x, y + d), (x + d, y + d)$, where $d \neq 0$), then $|A| = o(N^2)$.

D. PSEUDORANDOM GRAPHS

- ps4 D1. Let q be a prime. Let $S \subset \mathbb{F}_q \cup \{\infty\}$. Construct a graph G on vertex set \mathbb{F}_q^2 where two points are joined if the slope of the line connecting them lies in S . Viewed as a sequence of graphs as $q \rightarrow \infty$, prove that G is quasirandom as long as $|S|/q$ converges to a limit.
- ps4* D2. *Quasirandomness through fixed sized subsets.* Fix $p \in [0, 1]$. Let (G_n) be a sequence with $v(G_n) = n$. Write $G = G_n$.

(a) Fix a single $\alpha \in (0, 1)$. Suppose

$$e(S) = \frac{p\alpha^2 n^2}{2} + o(n^2) \quad \text{for all } S \subset V(G) \text{ with } |S| = \lfloor \alpha n \rfloor.$$

Prove that G is quasirandom.

(b) Fix a single $\alpha \in (0, 1/2)$. Write $\bar{S} = V(G) \setminus S$. Suppose

$$e(S, \bar{S}) = p\alpha(1 - \alpha)n^2 + o(n^2) \quad \text{for all } S \subset V(G) \text{ with } |S| = \lfloor \alpha n \rfloor.$$

Prove that G is quasirandom. Furthermore, show that the conclusion is false for $\alpha = 1/2$.

- D3. *Quasirandomness and regularity partitions.* Fix $p \in [0, 1]$. Let (G_n) be a sequence of graphs with $v(G_n) \rightarrow \infty$. Suppose that for every $\epsilon > 0$, there exists $M = M(\epsilon)$ so that each G_n has an ϵ -regular partition where all but ϵ -fraction of vertex pairs lie between pairs of parts with edge density $p + o(1)$ (as $n \rightarrow \infty$). Prove that G_n is quasirandom.

- ps4* D4. *Triangle counts on induced subgraphs.* Fix $p \in (0, 1]$. Let (G_n) be a sequence of graphs with $v(G_n) = n$. Let $G = G_n$. Suppose that for every $S \subset V(G)$, the number of triangles in the induced subgraph $G[S]$ is $p^3 \binom{|S|}{3} + o(n^3)$. Prove that G is quasirandom.
- D5. Prove that there are constant $\beta, \epsilon > 0$ such that for every even positive integer n and real $p \geq n^{-\beta}$, if G is an n -vertex graph where every vertex has degree $(1 \pm \epsilon)pn$ (meaning within ϵpn of pn) and every pair of vertices has codegree $(1 \pm \epsilon)p^2n$, then G has a perfect matching.

The next two exercises ask you to prove *Cheeger's inequality*:

$$\kappa/2 \leq h \leq \sqrt{2d\kappa}$$

for every d -regular graph with *spectral gap* $\kappa = d - \lambda_2$ and *edge-expansion ratio*

$$h := \min_{\substack{S \subset V \\ 0 < |S| \leq |V|/2}} \frac{e_G(S, V \setminus S)}{|S|}.$$

- ps4 D6. *Spectral gap implies expansion.* Prove that every d -regular graph with spectral gap κ has edge-expansion ratio $\geq \kappa/2$.
- ps4 D7. *Expansion implies spectral gap.* Let $G = (V, E)$ be a connected d -regular graph with spectral gap κ . Let $x = (x_v)_{v \in V} \in \mathbb{R}^V$ be an eigenvector associated to the second largest eigenvalue $\lambda_2 = d - \kappa$ of the adjacency matrix of G . Assume that $x_v > 0$ on at most half of the vertex set (or else we replace x by $-x$). Let $y = (y_v)_{v \in V} \in \mathbb{R}^V$ be obtained from x by replacing all its negative coordinates by zero.

(a) Prove that

$$d - \frac{\langle y, Ay \rangle}{\langle y, y \rangle} \leq \kappa.$$

Hint: recall that $\lambda_2 x_v = \sum_{u \sim v} x_u$.

(b) Let

$$\Theta = \sum_{uv \in E} |y_u^2 - y_v^2|.$$

Prove that

$$\Theta^2 \leq 2d(d \langle y, y \rangle - \langle y, Ay \rangle) \langle y, y \rangle.$$

Hint: $y_u^2 - y_v^2 = (y_u - y_v)(y_u + y_v)$. Apply Cauchy-Schwarz.

(c) Relabel the vertex set V by $[n]$ so that $y_1 \geq y_2 \geq \dots \geq y_t > 0 = y_{t+1} = \dots = y_n$. Prove

$$\Theta = \sum_{k=1}^t (y_k^2 - y_{k+1}^2) e([k], [n] \setminus [k]).$$

(d) Prove that for some $1 \leq k \leq t$,

$$\frac{e([k], [n] \setminus [k])}{k} \leq \frac{\Theta}{\langle y, y \rangle}.$$

(e) Prove the G has edge-expansion ratio $\leq \sqrt{2d\kappa}$.

- ps4 D8. Prove that the diameter of an (n, d, λ) -graph is at most $\lceil \log n / \log(d/\lambda) \rceil$. (The *diameter* of a graph is the maximum distance between a pair of vertices.)

ps4* D9. *Counting cliques.* For each part below, prove that for every $\epsilon > 0$, there exists $\delta > 0$ such that the conclusion holds for every (n, d, λ) -graph G with $d = pn$.

(a) If $\lambda \leq \delta p^2 n$, then the number of triangles of G is within a $1 \pm \epsilon$ factor of $p^3 \binom{n}{3}$.

(b) If $\lambda \leq \delta p^3 n$, then the number of K_4 's in G is within a $1 \pm \epsilon$ factor of $p^6 \binom{n}{4}$.

ps4 D10. Let p be an odd prime and $A, B \subset \mathbb{Z}/p\mathbb{Z}$. Show that

$$\left| \sum_{a \in A} \sum_{b \in B} \left(\frac{a+b}{p} \right) \right| \leq \sqrt{p|A||B|}$$

where (a/p) is the Legendre symbol defined by

$$\left(\frac{a}{p} \right) = \begin{cases} 0 & \text{if } a \equiv 0 \pmod{p} \\ 1 & \text{if } a \text{ is a nonzero quadratic residue mod } p \\ -1 & \text{if } a \text{ is a quadratic nonresidue mod } p \end{cases}$$

D11. *No spectral gap if too few generators.* Prove that for every $\epsilon > 0$ there is some $c > 0$ such that for every $S \subset \mathbb{Z}/n\mathbb{Z}$ with $0 \notin S = -S$ and $|S| \leq c \log n$, the second largest eigenvalue of the adjacency matrix of $\text{Cay}(\mathbb{Z}/n\mathbb{Z}, S)$ is at least $(1 - \epsilon)|S|$.

ps4* D12. Let p be a prime and let S be a multiplicative subgroup of \mathbb{F}_p^\times . Suppose $-1 \in S$. Prove that all eigenvalues of the adjacency matrix of $\text{Cay}(\mathbb{Z}/p\mathbb{Z}, S)$, other than the top one, are at most \sqrt{p} in absolute value.

D13. *Growth and expansion in quasirandom groups.* Let Γ be a finite group with no non-trivial representations of dimension less than K . Let $X, Y, Z \subset \Gamma$. Suppose $|X||Y||Z| \geq |\Gamma|^3/K$. Then $XYZ = \Gamma$ (i.e., every element of Γ can be expressed as xyz for some $(x, y, z) \in X \times Y \times Z$).

ps4 D14. Prove that for every positive integer d and real $\epsilon > 0$, there is some constant $c > 0$ so that every n -vertex d -regular graph has at least cn eigenvalues greater than $2\sqrt{d-1} - \epsilon$.
(Full credit will be awarded for proving the weaker statement that $\geq cn$ eigenvalues have *absolute value* $> 2\sqrt{d-1} - \epsilon$.)

ps4* D15. Show that for every d and r , there is some $\epsilon > 0$ such that if G is a d -regular graph, and $S \subset V(G)$ is such that every vertex of G is within distance r of S , then the top eigenvalue of the adjacency matrix of $G - S$ (i.e., remove S and its incident edges from G) is at most $d - \epsilon$.

E. GRAPH LIMITS

E1. *Zero-one valued graphons.* Let W be a $\{0, 1\}$ -valued graphon. Suppose graphons W_n satisfy $\|W_n - W\|_{\square} \rightarrow 0$ as $n \rightarrow \infty$. Show that $\|W_n - W\|_1 \rightarrow 0$ as $n \rightarrow \infty$.

ps5 E2. Define $W: [0, 1]^2 \rightarrow \mathbb{R}$ by $W(x, y) = 2 \cos(2\pi(x - y))$. Let F be a graph. Show that $t(F, W)$ is the number of ways to orient all edges of F so that every vertex has the same number of incoming edges as outgoing edges.

ps5 E3. *Weak regularity decomposition.* The following exercise offers alternate approach to the weak regularity lemma. It gives an approximation of a graphon as a linear combination of $\leq \epsilon^{-2}$ indicator functions of boxes. The polynomial dependence of ϵ^{-2} is important for designing efficient approximation algorithms.

- (a) Let $\epsilon > 0$. Show that for every graphon W , there exist measurable $S_1, \dots, S_k, T_1, \dots, T_k \subseteq [0, 1]$ and reals $a_1, \dots, a_k \in \mathbb{R}$, with $k < \epsilon^{-2}$, such that

$$\left\| W - \sum_{i=1}^k a_i \mathbf{1}_{S_i \times T_i} \right\|_{\square} \leq \epsilon.$$

The rest of the exercise shows how to recover a regularity partition from the above approximation.

- (b) Show that the stepping operator is contractive with respect to the cut norm, in the sense that if $W: [0, 1]^2 \rightarrow \mathbb{R}$ is a measurable symmetric function, then $\|W_{\mathcal{P}}\|_{\square} \leq \|W\|_{\square}$.
- (c) Let \mathcal{P} be a partition of $[0, 1]$ into measurable sets. Let U be a graphon that is constant on $S \times T$ for each $S, T \in \mathcal{P}$. Show that for every graphon W , one has

$$\|W - W_{\mathcal{P}}\|_{\square} \leq 2\|W - U\|_{\square}.$$

- (d) Use (a) and (c) to give a different proof of the weak regularity lemma (with slightly worse bounds than the one given in class): show that for every $\epsilon > 0$ and every graphon W , there exists partition \mathcal{P} of $[0, 1]$ into $2^{O(1/\epsilon^2)}$ measurable sets such that $\|W - W_{\mathcal{P}}\|_{\square} \leq \epsilon$.

ps5★

- E4. *Second neighborhood distance.* Let W be a graphon. Define $\tau_{W,x}: [0, 1] \rightarrow [0, 1]$ by

$$\tau_{W,x}(z) = \int_{[0,1]} W(x, y)W(y, z) dy.$$

(This models the second neighborhood of x .) Let $0 < \epsilon < 1/2$. Prove that if a finite set $S \subset [0, 1]$ satisfies

$$\|\tau_{W,s} - \tau_{W,t}\|_1 > \epsilon \quad \text{for all distinct } s, t \in S,$$

then $|S| \leq (1/\epsilon)^{C/\epsilon^2}$, where C is some absolute constant.

- E5. Show that for every $0 < \epsilon < 1/2$, every graphon lies within cut distance at most ϵ from some graph on at most C^{1/ϵ^2} vertices, where C is some absolute constant.

ps5

- E6. *Inverse counting lemma.* Using the uniqueness of moments theorem, deduce that for every $\epsilon > 0$ there is some $\eta > 0$ and integer $k > 0$ such that if U and W are graphons with

$$|t(F, U) - t(F, W)| \leq \eta \quad \text{whenever } v(F) \leq k,$$

then $\delta_{\square}(U, W) \leq \epsilon$.

ps5★

- E7. *Generalized maximum cut.* For symmetric measurable functions $W, U: [0, 1]^2 \rightarrow \mathbb{R}$, define

$$\mathcal{C}(W, U) := \sup_{\phi} \langle W, U^{\phi} \rangle = \sup_{\phi} \int W(x, y)U(\phi(x), \phi(y)) dx dy,$$

where ϕ ranges over all invertible measure preserving maps $[0, 1] \rightarrow [0, 1]$. Extend the definition of $\mathcal{C}(\cdot, \cdot)$ to graphs by $\mathcal{C}(G, \cdot) := \mathcal{C}(W_G, \cdot)$, etc.

- (a) Is $\mathcal{C}(U, W)$ continuous jointly in (U, W) with respect to the cut norm? Is it continuous in U if W is held fixed?
- (b) (*Key part of the problem*) Show that if W_1 and W_2 are graphons such that $\mathcal{C}(W_1, U) = \mathcal{C}(W_2, U)$ for all graphons U , then $\delta_{\square}(W_1, W_2) = 0$.

- (c) Let G_1, G_2, \dots be a sequence of graphs such that $\mathcal{C}(G_n, U)$ converges as $n \rightarrow \infty$ for every graphon U . Show that G_1, G_2, \dots is convergent.
- (d) Can the hypothesis in (c) be replaced by “ $\mathcal{C}(G_n, H)$ converges as $n \rightarrow \infty$ for every graph H ”?
- E8. (a) Let G_1 and G_2 be two graphs such that $\text{hom}(F, G_1) = \text{hom}(F, G_2)$ for every graph F . Show that G_1 and G_2 are isomorphic.
- (b) Let G_1 and G_2 be two graphs such that $\text{hom}(G_1, H) = \text{hom}(G_2, H)$ for every graph H . Show that G_1 and G_2 are isomorphic.

F. GRAPH HOMOMORPHISM INEQUALITIES

Recall some definitions. A graph F is said to be

- *Sidorenko* if $t(F, W) \geq t(K_2, W)^{e(F)}$ for all graphons W ;
- *forcing* if every graphon W with $t(F, W) = t(K_2, W)^{e(F)}$ is a constant graphon;
- *common* if $t(F, W) + t(F, 1 - W) \geq 2^{-e(F)+1}$ for all graphons W .

- F1. *Tensor power trick.* Let F be a bipartite graph. Suppose there is some constant $c > 0$ such that

$$t(F, G) \geq ct(K_2, G)^{e(F)} \quad \text{for all graphs } G.$$

Show that F is Sidorenko.

- F2. Prove that C_6 is Sidorenko.

ps5

- F3. Prove that $Q_3 = \begin{array}{ccc} & \square & \\ & \diagdown \diagup & \\ & \square & \end{array}$ is Sidorenko.

ps5

- F4. Prove that K_4^- is common, where K_4^- is K_4 with one edge removed.

- F5. Prove that every forcing graph is bipartite and has at least one cycle.

- F6. Prove that every forcing graph is Sidorenko.

- F7. *Forcing and quasirandomness.* Show that a graph F is forcing if and only if for every constant $p \in [0, 1]$, every sequence of graphs $G = G_n$ with

$$t(K_2, G) = p + o(1) \quad \text{and} \quad t(F, G) = p^{e(F)} + o(1)$$

is quasirandom.

- F8. *Forcing and stability.* Show that a graph F is forcing if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that if a graph G satisfies $t(F, G) \leq t(K_2, G)^{e(F)} + \delta$, then $\delta_{\square}(G, p) \leq \epsilon$.

- F9. Prove that $K_{s,t}$ is forcing whenever $s, t \geq 2$.

- F10. *A lower bound on clique density.* Show that for every positive integer $r \geq 3$, and graphs G , writing $p = t(K_2, W)$,

$$t(K_r, W) \geq p(2p - 1)(3p - 2) \cdots ((r - 1)p - (r - 2)).$$

Note that this inequality is tight when W is the associated graphon of a clique.

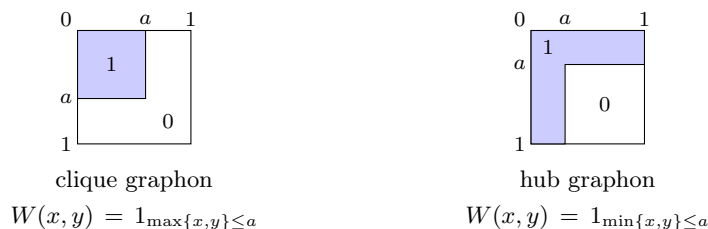
ps5★

- F11. Prove there is a function $f: [0, 1] \rightarrow [0, 1]$ with $f(x) \geq x^2$ and $\lim_{x \rightarrow 0} f(x)/x^2 = \infty$ such that

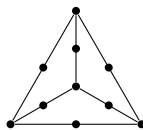
$$t(K_4^-, G) \geq f(t(K_3, G))$$

for all graphs G . Here K_4^- is K_4 with one edge removed.

- ps5** F12. *Cliquey edges.* Let n, r, t be nonnegative integers. Show that every n -vertex graph with at least $(1 - \frac{1}{r})\frac{n^2}{2} + t$ edges contains at least rt edges that belong to a K_{r+1} .
- ps5*** F13. *Maximizing $K_{1,2}$ density.* Prove that, for every $p \in [0, 1]$, among all graphons W with $t(K_2, W) = p$, the maximum possible value of $t(K_{1,2}, W)$ is attained by either a “clique” or a “hub” graphon, illustrated below.



- F14. Let F be the 3-graph with 10 vertices and 6 edges illustrated below (with the line segments denoting edges). Prove that the hypergraph Turán density of F is $2/9$.



G. FORBIDDING 3-TERM ARITHMETIC PROGRESSIONS

- ps5** G1. *Fourier uniformity does not control 4-AP counts.* Let

$$A = \{x \in \mathbb{F}_5^n : x \cdot x = 0\}.$$

Prove that:

- (a) $|A| = (5^{-1} + o(1))5^n$ and $|\widehat{1}_A(r)| = o(1)$ for all $r \neq 0$;
 (b) $|\{(x, y) \in \mathbb{F}_5^n : x, x + y, x + 2y, x + 3y \in A\}| \neq (5^{-4} + o(1))5^{2n}$.

- ps5*** G2. Fix $0 < \alpha < 1$. Let N be a prime. Let

$$A = \{x \in [N] : x^2 \bmod N < \alpha N\}.$$

Viewing $A \subset \mathbb{Z}/N\mathbb{Z}$, prove that, as $N \rightarrow \infty$ with fixed α ,

- (a) $|A| = (\alpha + o(1))N$ and $\max_{r \neq 0} |\widehat{1}_A(r)| = o(1)$;
 (b) $|\{(x, y) \in \mathbb{Z}/N\mathbb{Z} : x, x + y, x + 2y, x + 3y \in A\}| \neq (\alpha^4 + o(1))N^2$.

- ps6** G3. *Linearity testing.* Show that for every prime p there is some $C_p > 0$ such that if $f: \mathbb{F}_p^n \rightarrow \mathbb{F}_p$ satisfies

$$\mathbb{P}_{x, y \in \mathbb{F}_p^n} (f(x) + f(y) = f(x + y)) = 1 - \epsilon$$

then there exists some $a \in \mathbb{F}_p^n$ such that

$$\mathbb{P}_{x \in \mathbb{F}_p^n} (f(x) = a \cdot x) \geq 1 - C_p \epsilon.$$

In the above \mathbb{P} expressions x and y are chosen i.i.d. uniform from \mathbb{F}_p^n .

- ps6** G4. *Counting solutions to a single linear equation.*

(a) Given a function $f: \mathbb{Z} \rightarrow \mathbb{C}$ with finite support, define $\widehat{f}: \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{C}$ by

$$\widehat{f}(t) = \sum_{n \in \mathbb{Z}} f(n) e^{-2\pi i n t}.$$

Let $c_1, \dots, c_k \in \mathbb{Z}$. Let $A \subset \mathbb{Z}$ be a finite set. Show that

$$|\{(a_1, \dots, a_k) \in A^k : c_1 a_1 + \dots + c_k a_k = 0\}| = \int_0^1 \widehat{1}_A(c_1 t) \widehat{1}_A(c_2 t) \cdots \widehat{1}_A(c_k t) dt.$$

(b) Show that if a finite set A of integers contains $\beta |A|^2$ solutions $(a, b, c) \in A^3$ to $a + 2b = 3c$, then it contains at least $\beta^2 |A|^3$ solutions $(a, b, c, d) \in A^4$ to $a + b = c + d$.

ps6

G5. Let $a_1, \dots, a_m, b_1, \dots, b_m, c_1, \dots, c_m \in \mathbb{F}_2^n$. Suppose that the equation $a_i + b_j + c_k = 0$ holds if and only if $i = j = k$. Show that there is some constant $c > 0$ such that $m \leq (2 - c)^n$ for all sufficiently large n .

G6. *Sunflower-free subset.* Three sets A, B, C form a *sunflower* if $A \cap B = B \cap C = A \cap C = A \cap B \cap C$. Prove that there exists some $c > 0$ such that if \mathcal{F} is a collection of subsets of $[n]$ without a sunflower, then $|\mathcal{F}| \leq (3 - c)^n$ provided that n is sufficiently large.

G7. *Gowers U^2 uniformity norm.* Let $f: \mathbb{F}_p^n \rightarrow \mathbb{C}$, define

$$\|f\|_{U^2} := \left(\mathbb{E}_{x, h, h' \in \Gamma} f(x) \overline{f(x+h)} \overline{f(x+h')} f(x+h+h') \right)^{1/4}.$$

(a) Show that the expectation above is always a nonnegative real number, so that the above expression is well defined. Also, show that $\|f\|_{U^2} \geq |\mathbb{E}f|$.

(b) For $f_1, f_2, f_3, f_4: \mathbb{F}_p^n \rightarrow \mathbb{C}$, let

$$\langle f_1, f_2, f_3, f_4 \rangle = \mathbb{E}_{x, h, h' \in \Gamma} f_1(x) \overline{f_2(x+h)} \overline{f_3(x+h')} f_4(x+h+h').$$

Prove that

$$|\langle f_1, f_2, f_3, f_4 \rangle| \leq \|f_1\|_{U^2} \|f_2\|_{U^2} \|f_3\|_{U^2} \|f_4\|_{U^2}$$

(c) By noting that $\langle f_1, f_2, f_3, f_4 \rangle$ is multilinear, and using part (b), show that

$$\|f + g\|_{U^2} \leq \|f\|_{U^2} + \|g\|_{U^2}.$$

Conclude that $\|\cdot\|_{U^2}$ is a norm.

(d) Show that

$$\|f\|_{U^2} = \|\widehat{f}\|_{\ell^4}.$$

Furthermore, deduce that if $\|f\|_{\infty} \leq 1$, then

$$\|\widehat{f}\|_{\infty} \leq \|f\|_{U^2} \leq \|\widehat{f}\|_{\infty}^{1/2}.$$

(This gives a so-called “inverse theorem” for the U^2 norm: if $\|f\|_{U^2} \geq \delta$ then $|\widehat{f}(r)| \geq \delta^2$ for some $r \in \mathbb{F}_p^n$, i.e., if f is not U^2 -uniform, then it must correlate with some character.)

H. STRUCTURE OF SET ADDITION

ps6

H1. Show that for every real $K \geq 1$ there is some C_K such that for every finite set A of an abelian group with $|A + A| \leq K |A|$, one has $|nA| \leq n^{C_K} |A|$ for every positive integer n .

H2. Show that there is some constant C so that if S is a finite subset of an abelian group, and k is a positive integer, then $|2kS| \leq C^{|S|} |kS|$.

H3. Show that for every sufficiently large K there is there some finite set $A \subset \mathbb{Z}$ such that $|A + A| \leq K |A|$ and $|A - A| \geq K^{1.99} |A|$.

ps6★

H4. *Loomis–Whitney for sumsets.* Show that for every finite subsets A, B, C in an abelian group, one has

$$|A + B + C|^2 \leq |A + B| |A + C| |B + C|.$$

ps6★

H5. *Sumset versus difference set.* Let $A \subset \mathbb{Z}$. Prove that $|A - A|^{2/3} \leq |A + A| \leq |A - A|^{3/2}$.

ps6★

H6. *Another covering lemma.* Let A and B be finite sets in an abelian group satisfying $|A + A| \leq K |A|$ and $|A + B| \leq K' |B|$. Show that there exist some set X in the abelian group with $|X| = O(K \log(KK'))$ so that $A \subset \Sigma X + B - B$, where ΣX denotes the set of all elements that can be written as the sum of a subset of elements of X (including zero as the sum of the empty set).

Hint: Try first finding $2K$ disjoint translates $a + B$.

H7. *Modeling arbitrary sets of integers.* Let $A \subset \mathbb{Z}$ with $|A| = n$.

(a) Let p be a prime. Show that there is some integer t relatively prime to p such that

$$\|at/p\|_{\mathbb{R}/\mathbb{Z}} \leq p^{-1/n} \text{ for all } a \in A.$$

(b) Show that A is Freiman 2-isomorphic to a subset of $[N]$ for some $N = (4 + o(1))^n$.

(c) Show that (b) cannot be improved to $N = 2^{n-2}$.

(You may use the fact that the smallest prime larger than m has size $m + o(m)$.)

ps6

H8. *Sumset with 3-AP-free set.* Let A and B be n -element subsets of the integers. Suppose A is 3-AP free. Prove that $|A + B| \geq n(\log \log n)^{1/100}$ provided that n is sufficiently large.

Hint: Ruzsa triangle inequality, Plünnecke's inequality, Ruzsa model lemma, and Roth's theorem. Leave Freiman's theorem alone.

H9. *3-AP-free subsets of arbitrary sets of integers.* Prove that there is some constant $C > 0$ so that every set of n integers has a 3-AP-free subset of size at least $ne^{-C\sqrt{\log n}}$.

ps6

H10. *Bogolyubov with 3-fold sums.* Let $A \subset \mathbb{F}_p^n$ with $|A| = \alpha p^n$. Prove that $A + A + A$ contains a translate of a subspace of codimension $O(\alpha^{-3})$.

ps6★

H11. *Slightly better bounds on Bogolyubov.* Let $A \subset \mathbb{F}_2^n$ with $|A| = \alpha 2^n$.

(a) Show that if $|A + A| < 0.99 \cdot 2^n$, then there is some $r \in \widehat{\mathbb{F}_2^n} \setminus \{0\}$ such that $|\widehat{1_A}(r)| > c\alpha^{3/2}$ for some constant $c > 0$.

(b) By iterating (a), show that $A + A$ contains 99% of a subspace of codimension $O(\alpha^{-1/2})$.

(c) Deduce that $4A$ contains a subspace of codimension $O(\alpha^{-1/2})$ (i.e., Bogolyubov's lemma with better bounds than the one shown in class)