18.S997 (FALL 2017) PROBLEM SET 4

1. In class we showed that "Fourier controls 3-AP counts". In this problem, you will work out an example showing that "Fourier does not control 4-AP counts".
   Let $A = \{x \in \mathbb{F}_5^n : x \cdot x = 0\}$.[1] Write $N = 5^n$.
   (a) Show that $|A| = (1/5 + o(1))N$ and $|\widehat{1_A}(r)| = o(1)$ for all $r \neq 0$.
   (b) Show that $|\{(x, y) \in \mathbb{F}_5^n : x, x + y, x + 2y \in A\}| = (5^{-3} + o(1))N^2$.
   (c) Show that $|\{(x, y) \in \mathbb{F}_5^n : x, x + y, x + 2y, x + 3y \in A\}| = (5^{-3} + o(1))N^2$ (in particular, it is *not* $(5^{-4} + o(1))N^2$, which would be the case for a random subset A of density $1/5$).
2. Let Γ be a finite abelian group. Define, for $f : \Gamma \to \mathbb{C}$,
   $$\|f\|_{U^2} := \left(\mathbb{E}_{x,h,h' \in \Gamma} f(x)\overline{f(x+h)f(x+h')}f(x+h+h')\right)^{1/4}.$$
   (a) Show that the expectation above is always a nonnegative real number, so that the above expression is well defined. Also, show that $\|f\|_{U^2} \geq |\mathbb{E}f|$.
   (b) For $f_1, f_2, f_3, f_4 : \Gamma \to \mathbb{C}$, let
   $$\langle f_1, f_2, f_3, f_4 \rangle = \mathbb{E}_{x,h,h' \in \Gamma} f_1(x)\overline{f_2(x+h)f_3(x+h')}f_4(x+h+h').$$
   Prove that
   $$|\langle f_1, f_2, f_3, f_4 \rangle| \leq \|f_1\|_{U^2} \|f_2\|_{U^2} \|f_3\|_{U^2} \|f_4\|_{U^2}$$
   (c) By noting that $\langle f_1, f_2, f_3, f_4 \rangle$ is multilinear, and using part (b), show that
   $$\|f + g\|_{U^2} \leq \|f\|_{U^2} + \|g\|_{U^2}.$$
   Conclude that $\| \ \|_{U^2}$ is a norm.
   (d) Show that $\|f\|_{U^2} = \|\widehat{f}\|_{\ell^4}$, i.e., (it gives a different way of showing that $\| \ \|_{U^2}$ is a norm)
   $$\|f\|_{U^2}^4 = \sum_{\gamma \in \widehat{\Gamma}} |\widehat{f}(\gamma)|^4.$$
   Furthermore, deduce that if $\|f\|_\infty \leq 1$, then
   $$\|\widehat{f}\|_\infty \leq \|f\|_{U^2} \leq \|\widehat{f}\|_\infty^{1/2}.$$
   (This gives a so-called "inverse theorem" for the U^2 norm: if $\|f\|_{U^2} \geq \delta$ then $|f(\gamma)| \geq \delta^2$ for some $\gamma \in \widehat{\Gamma}$, i.e., if f is not U^2-uniform, then it must correlate with some character.)
3. Let $x_1, \ldots, x_m, y_1, \ldots, y_m, z_1, \ldots, z_m \in \mathbb{F}_2^n$. Suppose that the equation $x_i + y_j + z_k = 0$ holds if and only if $i = j = k$. Show that there is some constant $0 < C < 2$ such that $m \leq C^n$ for all sufficiently large n.
4. Show that for every finite subsets A, B, C in an abelian group, one has
   $$|A + B + C|^2 \leq |A + B|\,|A + C|\,|B + C|\,.$$

[1] Why $\mathbb{F}_5$?

5. For every real $K \geq 1$ and integer $n \geq 1$, let $f(n, K)$ be the minimum real number so that for every finite set A in an abelian group with $|A + A| \leq K|A|$, one has $|nA| \leq f(n, K)\,|A|$. For example, Plünneck–Ruzsa inequality gives $f(n, K) \leq K^n$.

   Show that[2], for every K there is some $C = C(K)$ such that $f(n, K) \leq n^C$ for all $n \geq 1$.
6. Let $A \subset \mathbb{Z}$ with $|A| = n$.
   (a) Let p be a prime. Show that there is some integer t relatively prime to p such that $\|at/p\|_{\mathbb{R}/\mathbb{Z}} \leq p^{-1/n}$ for all $a \in A$.
   (b) Show that A is Freiman 2-isomorphic to a subset of $[N]$ for some $N = (4 + o(1))^n$.
   (c) Show that (b) cannot be improved to $N = 2^{n-2}$.

   (You are allowed to use the fact that the smallest prime larger than m has size $m + o(m)$.)
7. Let $A \subset \mathbb{F}_2^n$ with $|A| = \alpha 2^n$.
   (a) Show that if $|A + A| < 0.99 \cdot 2^n$, then there is some $r \in \mathbb{F}_2^n \setminus \{0\}$ such that $|\widehat{1_A}(r)| > c\alpha^{3/2}$ for some constant $c > 0$.
   (b) By iterating (a), show that $A + A$ contains 99% of a subspace of codimension $O(\alpha^{-1/2})$. Deduce a version of Bogolyubov's lemma for $\mathbb{F}_2^n$ with better bounds: $4A$ contains a subspace of codimension $O(\alpha^{-1/2})$.

Problem set complete. Some hints on next page.
Multipart problems are worth 1/2 point per part.

[2] If a function $f(x, y)$ is bounded by a polynomial in x for each fixed y, and also bounded by a polynomial in y for each fixed x, is it necessarily bounded by a polynomial in x and y? Why?

Hints

1. Gauss sum
2. It may help to reparameterize the expression of the expectation so that the four arguments play more symmetric roles.
   Apply Cauchy–Schwarz generously.
   (Does it remind you of the calculations we did for C_4?)
3. Polynomial method
4. Either by using the Loomis-Whitney projection inequality or by induction on the sizes of the sets.
7. Should be reminiscent of the proof of Roth's theorem.