

18.S997 (FALL 2017) PROBLEM SET 3

1. Fix $0 < p < 1$. Let G be a graph on n vertices with average degree at least pn . Prove:
 - (a) The number of labeled 6-cycles in G is at least $(p^6 - o(1))n^6$.
 - (b) The number of labeled copies of $K_{3,3}$ in G is at least $(p^9 - o(1))n^6$.
 - (c) The number of labeled copies of $Q_3 = \begin{array}{c} \bullet & \bullet \\ \diagdown & \diagup \\ \bullet & \bullet \\ \diagup & \diagdown \\ \bullet & \bullet \end{array}$ in G is at least $(p^{12} - o(1))n^8$.
 - (d) (Bonus) The number of labeled paths on 4 vertices in G is at least $(p^3 - o(1))n^4$.
2. Deduce from the quasirandom Cayley graphs theorem the following corollary for vertex transitive graphs: If an n -vertex d -regular vertex-transitive graph G satisfies

$$|e(X, Y) - \frac{d}{n}|X||Y|| \leq \epsilon dn \quad \text{for all } X, Y \subseteq V(G),$$

then all the eigenvalues of the adjacency matrix of G , other than the largest one, are at most $8\epsilon d$ in absolute value.

3. Define $W: [0, 1]^2 \rightarrow \mathbb{R}$ by $W(x, y) = 2 \cos(2\pi(x - y))$. Let G be a graph. Show that $t(G, W)$ is the number of ways to orient all edges of G so that every vertex has the same number of incoming edges as outgoing edges.
4. Let W be a $\{0, 1\}$ -valued graphon. Suppose graphons W_n satisfy $\|W_n - W\|_{\square} \rightarrow 0$ as $n \rightarrow \infty$. Show that $\|W_n - W\|_1 \rightarrow 0$ as $n \rightarrow \infty$.
5. (a) Let $\epsilon > 0$. Show that for every graphon $W: [0, 1]^2 \rightarrow [0, 1]$, there exist measurable sets $S_1, \dots, S_k, T_1, \dots, T_k \subseteq [0, 1]$ and reals $a_1, \dots, a_k \in \mathbb{R}$, with $k < 1/\epsilon^2$, such that

$$\left\| W - \sum_{i=1}^k a_i \mathbf{1}_{S_i \times T_i} \right\|_{\square} \leq \epsilon.$$

- (b) Let \mathcal{P} be a partition of $[0, 1]$ into measurable sets. Let U be a graphon that is constant on $S \times T$ for each $S, T \in \mathcal{P}$. For that for every graphon W , one has

$$\|W - W_{\mathcal{P}}\|_{\square} \leq 2\|W - U\|_{\square}.$$

- (c) Use (a) and (b) to give a different proof of the weak regularity lemma (with slightly worse bounds than the one given in class): show that for every $\epsilon > 0$ and every graphon W , there exists partition \mathcal{P} of $[0, 1]$ into $2^{O(1/\epsilon^2)}$ measurable sets such that $\|W - W_{\mathcal{P}}\|_{\square} \leq \epsilon$.
6. In this problem, you will give an alternate proof of the strong regularity lemma with explicit bounds.

Let $\epsilon = (\epsilon_1, \epsilon_2, \dots)$ be a sequence of positive reals. By repeatedly applying the weak regularity lemma, show that there is some $M = M(\epsilon)$ such that for every graphon W , there is a pair of partitions \mathcal{P} and \mathcal{Q} of $[0, 1]$ into measurable sets, such that \mathcal{Q} refines \mathcal{P} , $|\mathcal{Q}| \leq M$ (here $|\mathcal{Q}|$ denotes the number of parts of \mathcal{Q}),

$$\|W - W_{\mathcal{Q}}\|_{\square} \leq \epsilon_{|\mathcal{P}|} \quad \text{and} \quad \|W_{\mathcal{Q}}\|_2^2 \leq \|W_{\mathcal{P}}\|_2^2 + \epsilon_1^2.$$

Furthermore, deduce the strong regularity lemma in the form given in class: one can write

$$W = W_{\text{str}} + W_{\text{psr}} + W_{\text{sml}}$$

where W_{str} is a k -step-graphon with $k \leq M$, $\|W_{\text{psr}}\|_{\square} \leq \epsilon_k$, and $\|W_{\text{sml}}\|_1 \leq \epsilon_1$. State your bounds¹ on M explicitly in terms of ϵ .

7. (Generalized maximum cut) For symmetric measurable functions $W, U: [0, 1]^2 \rightarrow \mathbb{R}$, define

$$\mathcal{C}(W, U) := \sup_{\varphi} \langle W, U^{\varphi} \rangle = \sup_{\varphi} \int W(x, y) U(\varphi(x), \varphi(y)) dx dy,$$

where φ ranges over all measure-preserving bijections on $[0, 1]$. Extend the definition of $\mathcal{C}(\cdot, \cdot)$ to graphs: $\mathcal{C}(G, \cdot) := \mathcal{C}(W_G, \cdot)$ etc.

- (a) Show that if W_1 and W_2 are graphons such that $\mathcal{C}(W_1, U) = \mathcal{C}(W_2, U)$ for all graphons U , then $\delta_{\square}(W_1, W_2) = 0$.
- (b) Let G_1, G_2, \dots be a sequence of graphs such that $\mathcal{C}(G_n, U)$ converges as $n \rightarrow \infty$ for every graphon U . Show that G_1, G_2, \dots is convergent.
- (c) Can the hypothesis in (b) be replaced by “ $\mathcal{C}(G_n, H)$ converges as $n \rightarrow \infty$ for every graph H ”?
8. Using the moments lemma ($t(F, U) = t(F, W)$ for all F implies $\delta_{\square}(U, W) = 0$) and compactness of the space of graphons, deduce:

Inverse counting lemma. For every $\epsilon > 0$, there exist $k \in \mathbb{N}$ and $\eta > 0$ such that whenever two graphons U and W satisfy

$$|t(F, U) - t(F, W)| \leq \eta \quad \text{for all graphs } F \text{ on } k \text{ vertices,}$$

we must have $\delta_{\square}(U, W) \leq \epsilon$.

9. (a) Given a function $f: \mathbb{Z} \rightarrow \mathbb{C}$ with finite support, define $\widehat{f}: \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{C}$ by

$$\widehat{f}(t) = \sum_{n \in \mathbb{Z}} f(n) e^{-2\pi i n t}.$$

Let $c_1, \dots, c_k \in \mathbb{Z}$. Let $A \subset \mathbb{Z}$ be a finite set. Show that

$$|\{(a_1, \dots, a_k) \in A^k : c_1 a_1 + \dots + c_k a_k = 0\}| = \int_0^1 \widehat{1}_A(c_1 t) \widehat{1}_A(c_2 t) \cdots \widehat{1}_A(c_k t) dt.$$

- (b) Show that if a finite set A of integers contains $\beta |A|^2$ solutions $(a, b, c) \in A^3$ to $a + 2b = 3c$, then it contains at least $\beta^2 |A|^3$ solutions $(a, b, c, d) \in A^4$ to $a + b = c + d$.

Problems 1 and 7 are worth 0.5 points per part

Problem set complete. Some hints on next page

¹With $\epsilon_k = \epsilon/k^2$ (corresponding to Szemerédi’s regularity lemma), your bound on M should be an exponential tower of 2’s of height $\epsilon^{-O(1)}$; if not then you are doing something wrong.

HINTS

4. Every measurable set can be arbitrarily well approximated (in measure) as a union of boxes.
7. Remember that $\|\cdot\|_{\square} \leq \|\cdot\|_1 \leq \|\cdot\|_2$.