TECHNIQUES FOR PROBLEMS ON SUMS AND INTEGRALS

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Contents

The goal of this lecture is to try to introduce some common techniques used to solve problems involving sums and integrals. Most of the material is adapted from an earlier lecture on the same topic by Evan Chen available at: https://yufeizhao.com/a34/fa18/sums-integrals.pdf.

1 Simplification by Grouping

Often times, there is some structured way to group the terms in a summation or integral which leads to drastic simplification.

1.1 Oddness

The most common use of this strategy is through taking advantage of the underlying oddness of the object of summation.

Example 1. Compute

$$\int_{-1}^{1} \frac{\cos x}{e^x + 1} \mathrm{d}x.$$

Proof. Observe that

$$\frac{\cos x}{e^x + 1} + \frac{\cos(-x)}{e^{-x} + 1} = \cos x \cdot \left(\frac{1}{e^x + 1} + \frac{e^x}{1 + e^x} \right) = \cos x.$$

It follows that

$$\int_{-1}^{1} \frac{\cos x}{e^x + 1} \mathrm{d}x = \int_{0}^{1} \cos x \mathrm{d}x = \sin 1.$$

$\square$

More generally, if we have a function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = f^+(x) + f^-(x)$, where the "even" component f^+ satisfies $f^+(x) = f^+(-x)$ for all $x \in \mathbb{R}$ and the "odd" component f^- satisfies $f^-(x) = -f^-(-x)$ for all $x \in \mathbb{R}$, then we have $\int_{-t}^{t} f(x)\mathrm{d}x = \int_{-t}^{t} f^+(x)\mathrm{d}x$. We can also write any function $f : \mathbb{R} \to \mathbb{R}$ uniquely in this form by taking $f^+(x) := \frac{f(x)+f(-x)}{2}$ and $f^-(x) := \frac{f(x)-f(-x)}{2}$.

1.2 Other Symmetry

Other times, there is some other form of symmetry which is less obvious at first glance.

Example 2 (2020 B4). Let n be a positive integer, and let V_n be the set of integer $(2n+1)$-tuples $\mathbf{v} = (s_0, s_1, \cdots, s_{2n-1}, s_{2n})$ for which $s_0 = s_{2n} = 0$ and $|s_j - s_{j-1}| = 1$ for $j = 1, 2, \cdots, 2n$. Define

$$q(\mathbf{v}) = 1 + \sum_{j=1}^{2n-1} 3^{s_j},$$

and let $M(n)$ be the average of $\frac{1}{q(\mathbf{v})}$ over all $\mathbf{v} \in V_n$. Evaluate $M(2020)$.

Proof. The key idea is to group together "cyclic shifts." In particular, we can define an equivalence relation $\sim$ via $(s_0, s_1, \cdots, s_{2n-1}, s_{2n}) \sim (s_i - s_i, s_{i+1} - s_i, s_{i+2} - s_i, \cdots, s_{i+2n} - s_i)$ where indices are modulo $2n$. We claim that the average value is $\frac{1}{4040}$ over each induced equivalence class. Indeed, the average value over the equivalence class containing $(s_0, s_1, \cdots, s_{2n})$ would be:

$$\frac{1}{4040} \sum_{i=0}^{2n-1} \frac{1}{\sum_{j=0}^{2n-1} 3^{s_j - s_i}} = \frac{1}{4040} \sum_{i=0}^{2n-1} \frac{3^{s_i}}{\sum_{j=0}^{2n-1} 3^{s_j}} = \frac{1}{4040}.$$

This yields $M(2020) = \frac{1}{4040}$.

□

2 Swapping

2.1 Finite Sums

We can of course freely swap the orders of finite summations due to commutativity of addition.

Example 3. Prove that

$$\sum_{k=1}^{n} \varphi(k) \left\lfloor \frac{n}{k} \right\rfloor = \frac{1}{2} n(n+1).$$

Proof. We first need to introduce an additional summation term into order to make the given summation easier to evaluate. Noting that $\left\lfloor \frac{n}{k} \right\rfloor = \sum_{k|m, m\le n} 1$, we can rewrite the given LHS and swap the summations:

$$\sum_{k=1}^{n} \varphi(k) \left\lfloor \frac{n}{k} \right\rfloor = \sum_{k=1}^{n} \varphi(k) \sum_{k|m, m\le n} 1 = \sum_{m\le n} \sum_{k|m} \varphi(k).$$

It's well-known that the inner sum is simply m, from which the desired equality follows.

□

2.2 Infinite Sums and Integrals

It turns out that under certain conditions (which are very important to check), we can usually perform similar manipulations over infinite summations and integrals.

Example 4 (2016 B6). Evaluate

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{n=0}^{\infty} \frac{1}{k2^n + 1}.$$

Proof. We can first rewrite the sum as

$$\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{k-1}}{k(k2^n+1)}.$$

Then, the key idea is to group the terms according to the value of $d = k2^n + 1$. Namely, we can rearrange the terms so that our summation becomes:

$$\sum_{d=2}^{\infty}\frac{1}{d}\sum_{k=\frac{d-1}{2^n}\in\mathbb{Z}}\frac{(-1)^{k-1}}{k}.$$

The inner sum is not too hard to evaluate. If we write $d-1 = 2^r m$ where m is odd, then the inner sum is precisely:

$$\frac{1}{m}-\frac{1}{2m}-\frac{1}{4m}-\cdots-\frac{1}{2^r m}=\frac{1}{m}\left(1-\frac{1}{2}-\frac{1}{4}-\cdots-\frac{1}{2^r}\right)=\frac{1}{2^r m}=\frac{1}{d-1}.$$

It follows that the sum in question becomes

$$\sum_{d=2}^{\infty}\frac{1}{d(d-1)}=\sum_{d=2}^{\infty}\left(\frac{1}{d-1}-\frac{1}{d}\right)=1,$$

after applying a well-known telescoping trick.

□

Our proof above is not quite complete, for reasons we will discuss next.

2.3 Convergence Guarantees

As mentioned earlier, there are certain conditions we need to check in order to conclude that we can reorder an infinite sum and obtain the same result. As an example of why this is **extremely important**, you can check that:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}=\ln(2)\neq\left(1-\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{6}-\frac{1}{8}\right)+\left(\frac{1}{5}-\frac{1}{10}-\frac{1}{12}\right)\cdots=\frac{\ln(2)}{2}.$$

Fortunately, this is not too hard most of the time.

Here we'll just go through the most commonly used conditions to check that we are allowed to rearrange stuff.

Definition 5. *A convergent series $\sum a_k$ is* **absolutely convergent** *if $\sum |a_k| < \infty$ (i.e. it converges), and otherwise it is* **conditionally convergent**.

Theorem 6 (Rearrangement okay iff absolutely convergent)**.** *Let $\sum a_n$ be a convergent series of real numbers.*

- *If $\sum a_n$ is absolutely convergent, it still converges to the same limit under any permutation of terms. Same holds for complex series.*
- *If $\sum a_n$ is conditionally convergent, there is a permutation of the terms for which the sum converges to* 2023.

The theorem above also holds for doubly-indexed summations, where $\sum_{i,j} a_{i,j}$ is absolutely convergent iff $\sum_{i,j} |a_{i,j}|$ converges.

As a **very** useful corollary of the above result:

Corollary 7 (Tonelli)**.** *For nonnegative $a_{m,n}$, it holds that:*

$$\sum_{m,n} a_{m,n} = \sum_m \left(\sum_n a_{m,n} \right) = \sum_n \left(\sum_m a_{m,n} \right),$$

where it's possible that all three diverge.

In general, in summations and integrals where the summands/integrands are always of the same sign, you can proceed with any integration without reservations, i.e. **nonnegative** $\implies$ **euphoria**.

For a more rigorous overview of convergence conditions (in particular in the case of integrals) see Evan Chen's handout: https://yufeizhao.com/a34/fa18/sums-integrals.pdf.

3 Introducing Additional Sums and Integrals

3.1 Taylor Series

Common ones:

- $\frac{1}{1-x} = 1 + x + x^2 + \cdots$, for all $|x| < 1$
- $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$, for all $x \in \mathbb{R}$
- $\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$, for all $|x| < 1$
- $\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$ for all $|x| < 1$

As a related comment, often times there will be some fraction in some form $\frac{1}{1+\textit{something}}$ where the *something* is known to have absolute value greater than 1. In such cases, we cannot apply the geometric Taylor series directly, but it is often helpful to use the first Taylor series by noting that $\frac{1}{1+\textit{something}} = \frac{1}{\textit{something}} \cdot \frac{1}{1+\frac{1}{\textit{something}}}$, and applying the geometric Taylor series to remove the second fraction.

There is a nice theorem about Taylor series in general:

Theorem 8. *Let f be an analytic function. Within its radius of convergence, the Taylor series for f will converge absolutely for any x as a series of complex numbers.*

Example 9. Compute

$$\int_0^1 \log x \log(1-x) \mathrm{d}x.$$

Proof. Using our formula for $\log(1-x)$ above, we write:

$$\int_0^1 \log x \log(1-x) \mathrm{d}x = -\int_0^1 \log x \sum_{n \geq 1} \frac{x^n}{n} \mathrm{d}x = \sum_{n \geq 1} \frac{1}{n} \int_0^1 x^n \log x \mathrm{d}x,$$

where we have applied Tonelli to swap the $\sum$ and $\int$.

We have that:

$$\int_0^1 x^n \log x \mathrm{d}x = \left[\frac{1}{n+1}x^{n+1}\log x - \frac{x^{n+1}}{(n+1)^2}\right]_0^1$$

from Integration by Parts, which is equal to $-\frac{1}{(n+1)^2}$, and so our desired quantity is:

$$\sum_{n\geq 1}\frac{1}{n(n+1)^2} = \sum_{n\geq 1}\frac{1}{n+1}\left(\frac{1}{n}-\frac{1}{n+1}\right) = \sum_{n\geq 1}\left(\frac{1}{n}-\frac{1}{n+1}\right) - \frac{1}{(n+1)^2} = 1-(\zeta(2)-1) = 2-\frac{\pi^2}{6}.$$

$\square$

3.2 Eliminating Fractions

The following seemingly obvious statement is surprisingly useful.

Lemma 10. *For any real number $s > -1$ it holds that:*

$$\frac{1}{s+1} = \int_0^1 t^s \mathrm{d}t.$$

As a simple example, let's suppose we want to evaluate $\sum_{n=0}^{\infty}\frac{x^n}{n+1}$ for some $|x| < 1$. With our new trick, we have (after appropriately checking for absolute convergence):

$$\sum_{n=0}^{\infty}\frac{x^n}{n+1} = \sum_{n=0}^{\infty}\int_0^1 (xt)^n \mathrm{d}t = \int_0^1 \sum_{n=0}^{\infty}(xt)^n \mathrm{d}t = \int_0^1 \frac{1}{1-xt}\mathrm{d}t = \left[-\frac{\log(1-xt)}{x}\right]_{t=0}^1 = -\frac{\log(1-x)}{x},$$

which is one way to derive the Taylor Series from above.

3.3 Dummy Variables

Sometimes it is useful to introduce dummy variables (usually indicator variables which take values of 0 or 1) to make sums easier to manipulate.

Example 11. Show that if $r_1, \ldots, r_n$ are nonnegative reals and $x_1, \ldots, x_n$ are real numbers then

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\min(r_i, r_j)x_i x_j \geq 0.$$

Proof. Write $\mathbf{1}_t$ for the function $\mathbf{1}_t(x) := 1$ if $x \leq t$ and 0 else. Then, we have:

$$\min(r_i, r_j) = \int_0^{\infty}\mathbf{1}_{r_i}(t)\mathbf{1}_{r_j}(t)\mathrm{d}t,$$

which means our original expression can be rewritten as:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^{\infty}\mathbf{1}_{r_i}(t)\mathbf{1}_{r_j}(t)x_i x_j \mathrm{d}t = \int_0^{\infty}\left(\sum_{i=1}^{n}\mathbf{1}_{r_i}(t)x_i\right)^2 \mathrm{d}t \geq 0.$$

$\square$