PROBLEMS ON SUMS AND INTEGRALS

Concept Problems

1. (P+B 468) Compute

$$\lim_{n\to\infty}\left[\frac{1}{\sqrt{4n^2-1^2}}+\frac{1}{\sqrt{4n^2-2^2}}+\cdots+\frac{1}{\sqrt{4n^2-n^2}}\right].$$

2. Find

$$\lim_{n\to\infty}\frac{1}{n}\sum_{a=1}^{n}\sum_{b=1}^{n}\frac{a}{a^2+b^2}.$$

3. Show that $\int_{-\infty}^{\infty} e^{-x^2}\,dx=\sqrt{\pi}$.

♡ 4. Exhibit a sequence a_{ij} indexed by $\mathbb{Z}^2$ such that

$$\sum_i(\sum_j a_{ij})\neq\sum_j(\sum_i a_{ij})$$

with all sums converging.

5. Exhibit a smooth (i.e., infinitely differentiable) function $f:\mathbb{R}\to\mathbb{R}$ such that $f(x)=1$ when $x<0$ yet $f(x)=0$ when $x>1$. (This function is one of the most important gadgets in Analysis).

Putnam-Style Problems

6. Evaluate the improper integral

$$\int_0^1\frac{\log(1-x)}{x}\,dx.$$

♡ 7. Determine the value of the improper integral

$$\int_0^\infty\frac{x}{e^x-1}\,dx.$$

♡ 8. (2022 A4) Suppose that $X_1, X_2, \ldots$ are real numbers between 0 and 1 that are chosen independently and uniformly at random. Let $S=\sum_{i=1}^{k}X_i/2^i$, where k is the least positive integer such that $X_k<X_{k+1}$, or $k=\infty$ if there is no such integer. Find the expected value of S.

9. (1997 A3) Evaluate the following:

$$\int_0^\infty\left(x-\frac{x^3}{2}+\frac{x^5}{2\cdot4}-\frac{x^7}{2\cdot4\cdot6}+\cdots\right)\left(1+\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot4^2}+\frac{x^6}{2^2\cdot4^2\cdot6^2}+\cdots\right)dx.$$

♡ 10. Show that

$$\int_0^1 x^{-x}\,dx=\sum_{n\geq1}n^{-n}.$$

11. (2014 B2) Suppose that f is a function on the interval $[1,3]$ such that $-1 \leq f(x) \leq 1$ for all x and $\int_1^3 f(x)\,dx = 0$. Determine the largest possible value of

$$\int_1^3 \frac{f(x)}{x}\,dx.$$

12. (2020 A3) Let $a_0 = \pi/2$, and let $a_n = \sin(a_{n-1})$ for $n \geq 1$. Determine whether $a_1^2 + a_2^2 + a_3^2 + \cdots$ converges or not.

13. (P+B 472) Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and satisfy $f(x) \geq 1$ for all x. Suppose that

$$f(x)f(2x)\dots f(nx) \leq 2018n^{2019}$$

for every positive integer n and $x \in \mathbb{R}$. Must f be constant?

14. A rectangle in $\mathbb{R}^2$ is called *great* if either its width or height is an integer. Prove that if a rectangle X can be dissected into great rectangles, then the rectangle X is itself great.

15. (PUMaC 2017) Compute

$$\sum_{k\geq 0} \frac{2^k}{5^{2^k}+1}.$$

16. (P+B §3.2] Prove that

$$\lim_{n\to\infty} \left(\prod_{k=0}^{n} \binom{n}{k} \right)^{\frac{1}{n(n+1)}} = \sqrt{e}.$$

17. (2010 A6) Let $f\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a strictly decreasing continuous function such that $\lim_{x\to\infty} f(x) = 0$. Prove that

$$\int_0^\infty \frac{f(x) - f(x+1)}{f(x)}\,dx$$

diverges.

♡ 18. (Russian box problem) A rectangular prism X is contained within a rectangular prism Y.

(a) Is it possible the surface area of X exceeds that of Y?

(b) Is it possible the sum of the 12 side lengths of X exceeds that of Y?

19. (2019 B2) For all $n \geq 1$, let

$$a_n = \sum_{k=1}^{n-1} \frac{\sin \frac{2k-1}{2n}\pi}{\cos^2 \frac{k-1}{2n}\pi \cos^2 \frac{k}{2n}\pi}.$$

Find

$$\lim_{n\to\infty} \frac{a_n}{n^3}.$$

20. For $a, b, c > 0$ prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{4}{a+b} + \frac{4}{b+c} + \frac{4}{c+a} \geq \frac{12}{3a+b} + \frac{12}{3b+c} + \frac{12}{3c+a}.$$

21. (2004 B5)Evaluate

$$\lim_{x\to 1^-}\prod_{n\geq 0}\left(\frac{1+x^{n+1}}{1+x^n}\right)^{x^n}.$$

22. (2004 A6) Suppose that $f\colon [0,1]^2\to\mathbb{R}$ is continuous. Show that

$$\int_0^1\left(\int_0^1 f(x,y)dx\right)^2 dy+\int_0^1\left(\int_0^1 f(x,y)dy\right)^2 dx$$
$$\leq\left(\int_0^1\int_0^1 f(x,y)\,dx\,dy\right)^2+\int_0^1\int_0^1 [f(x,y)]^2\,dx\,dy.$$

23. (2015 B6) For each positive integer k, let $A(k)$ be the number of odd divisors of k in the interval $\left[1,\sqrt{2k}\right)$. Evaluate:

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{A(k)}{k}.$$

24. (2019 A4) (a) Let $f(x,y,z)$ be a continuous real-valued function on $\mathbb{R}^3$. Suppose that for every sphere S of radius 1, the integral of $f(x,y,z)$ over the surface of S equals 0. Must $f(x,y,z)$ be identically 0?

    (b) What if f is required to be smooth and of compact support?

25. (2020 A6) For a positive integer N, let

$$f_N(x)=\sum_{n=0}^{N}\frac{N+1/2-n}{(N+1)(2n+1)}\sin((2n+1)x).$$

    Determine the smallest constant M such that $f_N(x)\leq M$ for all N and all real x.

26. (2021 A4) Let

$$I(R)=\iint_{x^2+y^2\leq R^2}\left(\frac{1+2x^2}{1+x^4+6x^2y^2+y^4}-\frac{1+y^2}{2+x^4+y^4}\right)dxdy.$$

    Find

$$\lim_{R\to\infty} I(R),$$

    or show that this limit does not exist.

Problems related to more advanced topics

27. For positive integer n, let $p(n)$ be the number of ways to partition n into the sum of some positive integers, quotient permutation of the integers. The sequence $p(n)$ starts $1,2,3,5,5,7,11,\ldots$.

    (a) Prove that

$$\lim_{n\to\infty}\frac{\log p(n)}{\sqrt{n}}$$

    exists, and find its value.

(b)(Hard) Let the answer of (a) be C. Prove that

$$\lim_{n\to\infty} \frac{np(n)}{e^{C\sqrt{n}}}$$

exists.

(c)(Hard)Determine the value of the limit in (b).

28. Prove that if f is a continuous function on the unit circle S^1 parametrized by $\theta \in [0, 2\pi)$, θ_0 is an angle such that $\frac{\theta_0}{2\pi}$ is irrational, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} f(m\theta_0) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta.$$

Hint: Fourier expansion does not hold here, but can you do something similar?

29. Let $f(x)$ be a smooth function with *compact support* on $\mathbb{R}$, and $\hat{f}$ be its Fourier transform

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi x\xi i} dx.$$

Prove the Poisson Summation Formula

$$\sum_{n\in\mathbb{Z}} \hat{f}(n) = \sum_{n\in\mathbb{Z}} f(n).$$

Using this result, prove that for all $t > 0$

$$\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} = \frac{1}{\sqrt{t}} \sum_{n\in\mathbb{Z}} e^{-\frac{\pi n^2}{t}}.$$

This formula is useful in the analytic extension of the Riemann Zeta function.

30. Prove the Jacobi Triple Product Formula for $|q| < 1$

$$\prod_{m=1}^{\infty} (1 - q^{2m})(1 + \omega^2 q^{2m-1})(1 + \omega^{-2} q^{2m-1}) = \sum_{n=-\infty}^{\infty} \omega^{2n} q^{n^2}.$$

Using this, either find the number of integer solutions for $12345654321 = a^2 + b^2$, or exhibit the difference between the number of even and odd partitions of n(a partition $n = a_1 + \cdots + a_k$ is even iff k is even).

31. A smooth function $f : \mathbb{C} \to \mathbb{C}$ is *harmonic* if

$$\Delta f := (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) f = 0.$$

(1) Prove the Mean Value Property of Harmonic functions: for any $z \in \mathbb{C}$ and $r > 0$, we have

$$\int_0^{2\pi} f(z + e^{i\theta} r) d\theta = 2\pi f(z).$$

(2) Find all harmonic functions on $\mathbb{C}$ that is real and positive everywhere.

(3)(Hard) Harmonic functions can be similarly defined on any open subset of the plane. Find all harmonic functions on $\mathbb{C}\backslash\{0\}$ that is real and positive on its domain.

32. (Hard) Let

$$f(z) = \sum_{i=0}^{d} c_i z^i$$

be a polynomial with strictly positive coefficients($c_i > 0$ for all $0 \leq i \leq d$). Let $A_{n,k}$ be the coefficient of z^n in $f^k(z)$. Prove that there exists some $K > 0$(possibly depending on f) such that for all $k > K$ and $1 \leq n \leq dk - 1$, we have

$$A_{n,k}^2 \geq A_{n-1,k} A_{n+1,k}.$$