18.A34 PROBLEMS #5

51. [1] A person buys a 30-year $\$100,000$ mortgage at an annual rate of 8%. What is his or her monthly payment?

52. (a) [1] Person A chooses an integer between 0 and $2^{11} - 1$, inclusive. Person B tries to guess A's number by asking yes-no questions. What is the minimum number of questions needed to guarantee that B finds A's number? Can the questions all be chosen in advance in an elegant way?

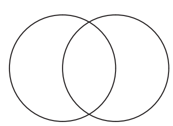
    (b) [2.5] What if A is allowed to lie at most once?

53. [1] Let M be an $n \times n$ *symmetric* matrix such that each row and column is a permutation of $1, 2, \ldots, n$. ("Symmetric" means that the entry in row i and column j is the same as the entry in row j and column i.) If n is odd, then show that every number $1, 2, \ldots, n$ appears exactly once on the main diagonal. For instance,

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \\ 3 & 1 & 5 & 2 & 4 \\ 4 & 5 & 2 & 3 & 1 \\ 5 & 3 & 4 & 1 & 2 \end{bmatrix}$$

54. [1] Find all 10 digit numbers $a_0 a_1 \cdots a_9$ such that a_i is the number of digits equal to i, for all $0 \le i \le 9$.

55. [1] Two circles of radius one pass through each other's centers. What is the area of their intersection?



56. (a) [2] Given any 1000 points in the plane, show that there is a circle which contains exactly 500 of the points in its interior, and none on its circumference.

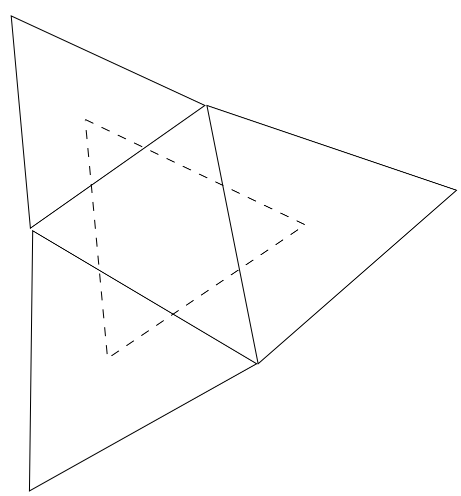
(b) [3] Given 1001 points in the plane, no three collinear and no four concyclic (i.e., no four on a circle), show that that there are exactly 250,000 circles with three of the points on the circumference, 499 points inside, and 499 points outside.

57. (a) [2.5] Let n be an integer, and suppose that $n^4 + n^3 + n^2 + n + 1$ is divisible by k. Show that either k or $k - 1$ is divisible by 5. HINT. First show that one may assume that k is prime. Use *Fermat's theorem* for the prime k, which states that if m is not divisible by k, then $m^{k-1} - 1$ is divisible by k. Try to avoid more sophisticated tools.

(b) [2] Deduce that there are infinitely many primes of the form $5j+1$.

58. [2] A cylindrical hole is drilled straight through and all the way through the center of a sphere. After the hole is drilled, its length is six inches. What is the volume that remains?

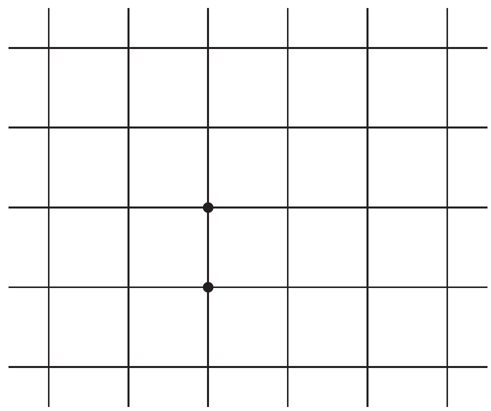
59. [2.5] Let T be a triangle. Erect an equilateral triangle on each side of T (facing outwards). Show that the centers of these equilateral triangles form the vertices of an equilateral triangle.



60. [5] Define a sequence $X_0, X_1, \ldots$ of rational numbers by $X_0 = 2$ and $X_{n+1} = X_n - \frac{1}{X_n}$ for $n \geq 0$. Is the sequence bounded?

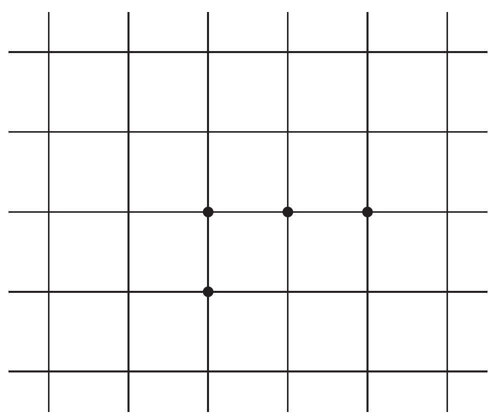
61. Let $B = \mathbb{Z} \times \mathbb{Z}$, regarded as an infinite chessboard. (Here $\mathbb{Z}$ denotes the set of integers.) Suppose that counters are placed on some subset of the points of B. A counter can jump over another counter one step vertically or horizontally to an empty point, and then remove the counter that was jumped over. Given $n > 0$, let $f(n)$ denote the least number of counters that can be placed on B such that all their y-coordinates are ≤ 0, and such that by some sequence of jumps it is possible for a counter to reach a point with y-coordinate equal to n. For instance, $f(1) = 2$, as shown by the following diagram.

$y = 0$



Similarly $f(2) = 4$, as shown by:

$y = 0$



(a) [2] Show that $f(3) = 8$ (or at least that $f(3) \leq 8$ by constructing a suitable example).

(b) [2.5] Show that $f(4) = 20$ (or at least that $f(4) \leq 20$).

(c) [3] Find an upper bound for $f(5)$.

62. [3.5] Generalize Problem 12 to n dimensions as follows. Show that there exist $n+1$ lattice points (i.e., points with integer coordinates) in $\mathbb{R}^n$ such that any two of them are the same distance apart if and only if n satisfies the following conditions:

   (a) If n is even, then $n+1$ is a square.

   (b) If $n \equiv 3 \pmod{4}$, then it is always possible.

   (c) If $n \equiv 1 \pmod{4}$, then $n+1$ is a sum of two squares (of nonnegative integers). The well-known condition for this is that if $n+1 = p_1^{a_1} \cdots p_r^{a_r}$ is the factorization of $n+1$ into prime powers, then a_i is even whenever $p_i \equiv 3 (\mathrm{mod}\ 4)$.

63. [5+] Let $H_n = \sum_{j=1}^{n} 1/j$. Show that for all $n \geq 1$,

$$\sum_{d|n} d \leq H_n + (\log H_n) e^{H_n}.$$