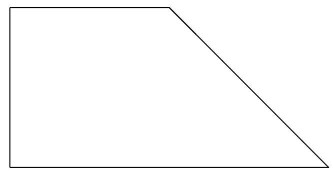
18.A34 PROBLEMS #10

111. [1] The following shape consists of a square and half of another square of the same size, divided diagonally.



Cut the shape into four congruent pieces.

112. [1.5] Take five right triangles with legs of length one and two, cut one of them, and put the resulting six pieces together to form a square.

113. [1.5] Find an integer n whose first digit is three, such that $3n/2$ is obtained by removing the 3 at the beginning and putting it at the end.

114. (a) [1] Show that for any real x, $e^x > x$.
(b) [1.5] Find the largest real number α for which it is *false* that $\alpha^x > x$ for all real x.

115. (a) [1] What amounts of postage cannot be obtained using only 5 cent stamps and 7 cent stamps? (For instance, 9 cents cannot be obtained, but $17 = 5 + 5 + 7$ cents can be.)
(b) [2.5] Let a and b be relatively prime positive integers. For how many positive integers c is it impossible to obtain postage of c cents using only a cent and b cent stamps? What is the largest value of c with this property?

116. [2.5] Two players A and B play the following game. Fix a positive real number x. A and B each choose the number 1 or 2. A gives B one dollar if the numbers are different. B gives A x dollars times the sum of their numbers. For instance, if A chooses 1 and B chooses 2, then A gives B one dollar and B gives A $3x$ dollars. Both players are playing their best possible strategy. What value of x makes the game fair, i.e., in the long run both players should break even?

117. [3] Given positive integers n and b, define the *total b-ary expansion* $T_b(n)$ as follows: Write n as a sum of powers of b, with no power occurring more than $b-1$ times. (This is just the usual base b expansion of n.) For instance, if $n = 357948$ and $b = 3$, then we get

$$3^{11} + 3^{11} + 3^7 + 3^6 + 3^6 + 3^2.$$

Now do the same for each exponent, giving

$$3^{3^2+1+1} + 3^{3^2+1+1} + 3^{3+3+1} + 3^{3+3} + 3^{3+3} + 3^{1+1}.$$

Continue doing the same for every exponent not already a b or 1, until finally only b's and 1's appear. In the present case we get that $T_3(357948)$ is the array

$$3^{3^{1+1}+1+1} + 3^{3^{1+1}+1+1} + 3^{3+3+1} + 3^{3+3} + 3^{3+3} + 3^{1+1}.$$

Now define a sequence $a_0, a_1, \ldots$ as follows. Choose a_0 to be any positive integer, and choose a base $b_0 > 1$. To get a_1, write the total b_0-ary expansion $T_{b_0}(a_0 - 1)$ of $a_0 - 1$, choose a base $b_1 > b_0$, and replace every appearance of b_0 in $T_{b_0}(a_0 - 1)$ by b_1. This gives the total b_1-ary expansion of the next term a_1. To get a_2, write the total b_1-ary expansion $T_{b_1}(a_1 - 1)$ of $a_1 - 1$, choose a base $b_2 > b_1$, and replace every appearance of b_1 in $T_{b_1}(a_1 - 1)$ by b_2. This gives the total b_2-ary expansion of the next term a_2. Continue in this way to obtain $a_3, a_4, \ldots$. In other words, given a_n and the previously chosen base b_n, o get a_{n+1}, write the total b_n-ary expansion $T_{b_n}(a_n - 1)$ of $a_n - 1$, choose a base $b_{n+1} > b_n$, and replace every appearance of b_n in $T_{b_n}(a_n - 1)$ by b_{n+1}. This gives the total b_{n+1}-ary expansion of the next term a_{n+1}.

Example. Choose $a_0 = 357948$ and $b_0 = 3$ as above. Then

$$a_0 - 1 = 357947 = 3^{3^{1+1}+1+1} + 3^{3^{1+1}+1+1} + 3^{3+3+1} + 3^{3+3} + 3^{3+3} + 3 + 3 + 1 + 1.$$

Choose $b_1 = 10$. Then

$$\begin{aligned} a_1 &= 10^{10^{1+1}+1+1} + 10^{10^{1+1}+1+1} + 10^{10+10+1} + 10^{10+10} + 10^{10+10} + 10 + 10 + 1 + 1 \\ &= 10^{102} + 10^{102} + 10^{21} + 10^{20} + 10^{20} + 22. \end{aligned}$$

Then

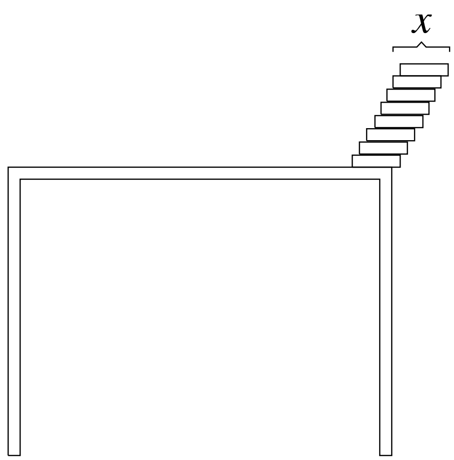
$$a_1 - 1 = 10^{10^{1+1}+1+1} + 10^{10^{1+1}+1+1} + 10^{10+10+1} + 10^{10+10} + 10^{10+10} + 10 + 10 + 1.$$

Choose $b_2 = 766$. Then

$$a_2 = 766^{766^{1+1}+1+1} + 766^{766^{1+1}+1+1} + 766^{766+766+1} + 766^{766+766} + 766^{766+766} + 766 + 766 + 1,$$

etc. Prove that for some n we have $a_n = a_{n+1} = \cdots = 0$. (Note how counterintuitive this seems. How could we not force $a_n \to \infty$ by choosing the b_n's sufficiently large?)

118. [2.5] What is the longest possible overhang x that can be obtained by stacking dominos of unit length over the edge of a table, as illustrated below? (The condition for the dominos not to fall is that the center of mass of all the dominos above any domino D lies directly above D.)

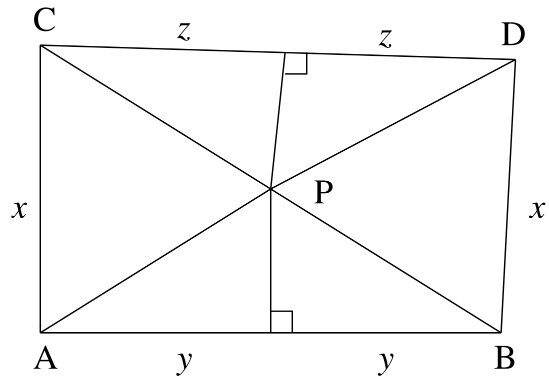


119. [3] Let G be a simple (i.e., no loops or multiple edges) finite graph and v a vertex of G. The *neighborhood* $N(v)$ of v consists of v and all adjacent vertices. Show that there exists a subset S of the vertex set of G such that $\#(S \cap N(v))$ is odd for all vertices v of G.

120. [2.5] Let G be a simple graph with an odd number of vertices. Show that there exists a nonempty subset S of the vertices such that every vertex of G is adjacent to an even number of elements of S. (A vertex is *not* considered to be adjacent to itself.)

121. [2.5] An ant is constrained to walk on the walls, floor, and ceiling of a $1 \times 1 \times 2$ room. The ant stands in a corner of the room. From its perspective, what point(s) in the room is the farthest away, and what is the distance of this point from the ant? Hint. The farthest point is *not* the opposite corner!

122. [2] Draw a line segment AB in the plane. Let AC be perpendicular to AB. Draw BD so that D is on the same side of AB as C, angle ABD is 91°, and AC and BD have the same length. Let the perpendicular bisectors of AB and CD meet in P. Angles PAB and PBA are equal since P is on the perpendicular bisector of AB. For the same reason AP and BP have the same length. Similarly CP and DP have the same length. Thus triangles ACP and BDP are congruent, so the angles CAP and DBP are equal. It follows that the angles CAB and DBA are equal, i.e., $90° = 91°$. Is there a flaw in this argument, or is this the end of mathematics?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