1 Induction

Recurrence is usually uniquely determined by the given recurrence relation and the initial condition. Therefore if we can "guess" a close form of a recurrence, then we can prove that this is indeed the answer by using induction.

Example 1.1. The Fibonacci numbers F_n satisfies the recurrence relation

$$F_{n+2} = F_{n+1} + F_n \quad \forall n \geq 0$$

and initial condition $F_0 = 0, F_1 = 1$. We can prove that

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

by induction. The base case when $n = 0, 1$ is easy to verify. If it holds for $n = k, k+1$, then

$$F_{k+2} = F_k + F_{k+1} = \frac{1}{\sqrt{5}}\alpha^k(\alpha+1) - \frac{1}{\sqrt{5}}\beta^k(\beta+1) = \frac{1}{\sqrt{5}}\alpha^{k+2} - \frac{1}{\sqrt{5}}\beta^{k+2}$$

where $\alpha, \beta = (1 \pm \sqrt{5})/2$ are both roots of $t^2 - t - 1 = 0$.

Using this idea, we can solve all the homogeneous linear recurrence.

Example 1.2. Suppose that we are given initial values $a_0, \ldots, a_{d-1}$ and also some constants $c_0, \ldots, c_{d-1}$. Consider the recurrence relation given by

$$a_{n+d} = c_{d-1}a_{n+d-1} + \cdots + c_0 a_n \quad \forall n \geq 0.$$

If the polynomial $t^d - c_{d-1}t^{d-1} - \cdots - c_1 t - c_0$ has roots $\alpha_1, \ldots, \alpha_k$ with multiplicities $m_1, \ldots, m_k$, then the sequence

$$\langle n^s \alpha_i^n \rangle_{n=0}^{\infty}$$

satisfies the recurrence relation as long as $s < m_i$. We then find $m_1 + \cdots + m_k = d$ sequences which satisfy the recurrence relation, and any linear combination of them also satisfies the recurrence. They are linearly independent, so we can actually find a linear combination that satisfies the initial value conditions. Now we can prove that the sequence is exactly this specific linear combination by induction.

2 Generating Function

For a sequence $a_0, a_1, \ldots$, the generating function $A(x)$ of it is defined as

$$A(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Note that this is just a formal series, meaning that this series needs not to converge at any neighborhood of 0. Therefore it does not make any sense to plug in a specific value of x, unless we know something else about the sequence. This is just a convenient way to write some complicated operation into a simpler way. For example, we can define the sum as

$$A(x)+B(x)=\sum_{n=0}^{\infty}(a_n+b_n)x^n$$

and the product as

$$A(x)B(x)=\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}a_ib_{n-i}\right)x^n.$$

The formal series actually form a commutative ring with 1 in this way. The multiplicative inverse does not necessarily exist.

Property 2.1. $A(x)^{-1}$ exists if and only if a_0 is nonzero.

Proof. If $A(x)^{-1}$ exists then we know that there exists $B(x)$ such that $A(x)B(x)=1$. In particular $a_0b_0=1$ and so $a_0\neq 0$.

On the other hand, if a_0 is nonzero, then we can set $b_0=a_0^{-1}$ and iteratively define

$$b_n=-a_0^{-1}\sum_{i=0}^{n-1}b_ia_{n-i}$$

for all $n\geq 1$. Then it is clear that $A(x)B(x)=1$. □

Now that we know we can safely divide any formal series with nonzero constant term, we can give another way to solve the linear recurrence.

Example 2.1. Let A be the generating function of Fibonacci numbers. Consider $B(x)=(1-x-x^2)A(x)$. Then

$$b_{n+2}=F_{n+2}-F_{n+1}-F_n=0$$

for all $n\geq 0$. Hence $B(x)=1$ and so

$$A(x)=\frac{x}{1-x-x^2}=\frac{P}{1-\alpha x}+\frac{Q}{1-\beta x}$$

where P,Q are some constant and $\alpha,\beta=(1\pm\sqrt{5})/2$. If we multiply $(1-\alpha x)$ on both side and plug in $x=\alpha^{-1}$ (this is valid since we can work in the field of rational function), then we get that

$$P=\frac{\alpha^{-1}}{1-\alpha^{-1}\beta}=\frac{1}{\sqrt{5}}$$

and similarly $Q=-\frac{1}{\sqrt{5}}$. Now note that

$$\frac{1}{1-\alpha x}=1+\alpha x+\alpha^2x^2+\cdots$$

and

$$\frac{1}{1-\beta x}=1+\beta x+\beta^2x^2+\cdots.$$

Therefore

$$F_n=\frac{1}{\sqrt{5}}\alpha^n-\frac{1}{\sqrt{5}}\beta^n.$$

Example 2.2. In general, if we have a linear recurrence

$$a_{n+d} = c_{d-1}a_{n+d-1} + \cdots + c_0 a_n$$

and A is the generating function of this sequence, then

$$A(x) = \frac{P(x)}{1 - c_{d-1}x - \cdots - c_0 x^d}$$

for some polynomial P of degree less than d. If the polynomial $t^d - c_{d-1}t^{d-1} - \cdots - c_0$ has roots $\alpha_1, \ldots, \alpha_k$ with multiplicities $m_1, \ldots, m_k$, then $A(x)$ is a linear combination of

$$\frac{1}{(1 - \alpha_i x)^s}$$

for $i = 1, \ldots, k$ and $s = 0, \ldots, m_i - 1$. Note that this has a convergent Taylor series

$$\frac{1}{(1 - \alpha_i x)^s} = \sum_{n=0}^{\infty} \binom{-s}{n} (-\alpha)^n x^n = \sum_{n=0}^{\infty} \binom{n+s-1}{s-1} \alpha^n x^n.$$

Therefore the sequence $a_0, a_1, \ldots$ is a linear combination of the sequences

$$\langle \binom{n+s-1}{s-1} \alpha^n \rangle_{n=0}^{\infty},$$

which is equivalent to what we got before.

Example 2.3. The Catalan number satisfies the recurrence $C_0 = 1$ and

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i}.$$

We can also solve this by generating function: if we let $F(x)$ be the generating function of Catalan number, then

$$F(x) - xF(x)^2 = 1.$$

For a moment, let's ignore the question of whether it is valid to just solve this equation by the quadratic formula and just directly do so. Then we get that

$$F(x) = \frac{1 \pm \sqrt{1-4x}}{2x}.$$

We know that we should choose the minus sign so that the Taylor expansion of the right hand side exists at 0. Now $(1-\sqrt{1-4x})/2x$ is a function who satisfies $F(x) - xF(x)^2 = 1$ and takes value 1 at 0. Therefore its Taylor expansion should also satisfy this, and so $F(x) = (1 - \sqrt{1-4x})/2x$ by the uniqueness of F.

That's the only technicality, and the rest is just simple calculation. By binomial theorem,

$$\sqrt{1-4x} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-4)^n x^n = 1 - \sum_{n=0}^{\infty} 4 \binom{\frac{1}{2}}{n+1} (-4)^n x^{n+1} = 1 - 2x \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^n.$$

Hence $C_n = \binom{2n}{n}/(n+1)$.

Sometimes to do some operation, we need that the formal series to convergent. Usually, in order to do so, we consider the exponential generating function

$$A(x) = \sum_{n=0}^{\infty} \frac{1}{n!} a_n x^n.$$

Example 2.4. Consider the recurrence $a_{n+2} = (2n+7/2)a_{n+1} - (n^2+5n/2+3/2)a_n$. Let A be the exponential generating function of the sequence. Write $(2n+7/2) = 2(n+2)-1/2$ and $n^2 + 5n/2 + 3/2 = (n+1)(n+2) - (n+1)/2$. Therefore

$$[(1-2x+x^2)A(x)]' + \left(\frac{1}{2} - \frac{1}{2}x\right) A(x) = c$$

for some constant c. We can rewrite it as

$$\left[(1-x)^{\frac{3}{2}} A(x)\right]' = c\,(1-x)^{-\frac{1}{2}}.$$

Therefore there exists two constants c_1, c_2 such that

$$A(x) = \frac{c_1}{1-x} + \frac{c_2}{(1-x)^{\frac{3}{2}}}.$$

This shows that

$$\frac{a_n}{n!} = c_1 + (-1)^n c_2 \binom{-\frac{3}{2}}{n} = c_1 + c_2 \frac{(2n+1)!!}{2^n n!}$$

and so

$$a_n = c_1 \cdot n! + c_2 \cdot \frac{(2n+1)!!}{2^n}.$$

3 Pigeon Hole Principle

If there is a recurrence relation

$$a_n = f(a_{n-1}, \ldots, a_{n-d})$$

where f only takes finitely many values, then a_n is eventually periodic. This is because that if f can only take value in a finite set S, then $(a_{n-1}, \ldots, a_{n-d})$ can only be in S^d eventually, and so by pigeon hole principle there should be m, n such that $a_{m-i} = a_{n-i}$ where $i = 1, \ldots, d$. Then we can prove by induction that $a_{m+t} = a_{n+t}$ for all $t \geq 0$ by induction.

Example 3.1. Integral linear recurrence mod n is eventually periodic.

4 Backward Recurrence

In the previous setting,

$$a_n = f(a_{n-1}, \ldots, a_{n-d})$$

and $a_i \in S$ where S is a finite set. If there is an "inverse" function g such that

$$a_{n-d} = g(a_n, a_{n-1}, \ldots, a_{n-d+1}),$$

then we can extend the sequence $a_0, a_1, \ldots$ in the opposite direction with g, and get $\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots$. Since a_n is eventually periodic, it is periodic now. Therefore if we compute, say, a_{-1}, then we can tell that there exists $n > 0$ such that $a_n = a_{-1}$. This could be really useful sometimes.

Example 4.1. Let c be any integer, and $a_1 = 1, a_2 = c$. For all $n \geq 1$,

$$a_{n+2} = ca_{n+1} + a_n.$$

Then show that $a_m | a_n$ if $m | n$.

Solution. We can define a backward recurrence

$$a_n = a_{n+2} - ca_{n+1}$$

and then extend the sequence backward. We then find that $a_0 = 0$. Consequently, for any linear recurrence

$$b_{n+2} = cb_{n+1} + b_n$$

with $b_0 = 0, b_1 = 1$, we have that $b_m \equiv 0 \mod a_m$. Then it is not hard to see that when the initial condition is replaced by $b_0 \equiv 0 \mod a_m$ and b_1 being anything, we still have $b_m \equiv 0 \mod a_m$. This shows that if $a_m | a_i$ then $a_m | a_{i+m}$, and so we are done. ■

5 Variable Change

Sometimes recurrences can be easily solved after a variable change.

Example 5.1. Consider a sequence $a_1, a_2, \ldots$ with a recurrence

$$a_{n+1} = \frac{a_n}{a_n + 1}$$

for all $n \geq 1$. If we consider a variable change

$$b_n = \frac{a_n - \frac{1}{n}}{a_n},$$

then

$$b_{n+1} = \frac{a_{n+1} - \frac{1}{n+1}}{a_{n+1}} = \frac{n}{n+1} \frac{a_n - \frac{1}{n}}{a_n} = \frac{n}{n+1} b_n.$$

This recurrence is easy to solve: we can easily see that $b_n = b_1/n$. Therefore

$$a_n = \frac{1}{n(1 - b_n)} = \frac{1}{n - b_1} = \frac{a_1}{(n-1)a_1 + 1}.$$

Example 5.2. Consider another sequence $a_0, a_1, \ldots$ with a recurrence

$$a_{n+1} = \frac{2a_n + 1}{a_n + 2}$$

for all $n \geq 0$. If we consider a variable change

$$b_n = \frac{a_n + 1}{a_n - 1}$$

then

$$b_{n+1} = \frac{a_{n+1} + 1}{a_{n+1} - 1} = \frac{3(a_n + 1)}{a_n - 1} = 3b_n.$$

This recurrence is also easy to solve: $b_n = 3^n b_0$. Therefore

$$a_n = \frac{b_n + 1}{b_n - 1} = \frac{3^n b_0 + 1}{3^n b_0 - 1} = \frac{(3^n + 1)a_n + (3^n - 1)}{(3^n - 1)a_n + (3^n + 1)}.$$

In general, if we have a recurrence

$$a_{n+1} = \frac{pa_n + q}{ra_n + s}$$

and the equation

$$x = \frac{px + q}{rx + s}$$

has two distinct roots α, β, then we can consider a variable change

$$b_n = \frac{a_n - \alpha}{a_n - \beta}$$

and then we can get a recurrence of the form $b_{n+1} = cb_n$ for some constant c.

Example 5.3. Sometimes trigonometric functions are useful too. For example, if we are trying to solve

$$a_{n+1} = 1 - 2a_n^2,$$

and $a_0 = \cos\theta_0$, then $a_n = \cos 2^n\theta_0$. This helps us even when $|a_0| > 1$: we can simply look for $z \in \mathbb{C}$ such that $z + \frac{1}{z} = 2a_0$. This way, we can simply get that

$$a_n = \frac{1}{2}\left(z^{2^n} + \frac{1}{z^{2^n}}\right).$$