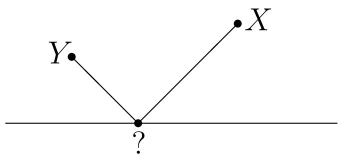
18.A34 PROBLEMS #4

40. [1] Three students A, B, C compete in a series of tests. For coming in first in a test, a student is awarded x points; for coming second, y points; for coming third, z points. Here x, y, z are positive integers with $x > y > z$. There were no ties in any of the tests. Altogether A accumulated 20 points, B 10 points, and C 9 points. Student A came in second in the algebra test. Who came in second in the geometry test?

41. [1] Remove the upper-left and lower-right corner squares from an 8×8 chessboard. Show that the resulting board cannot be covered by 31 dominoes. (A domino consists of two squares with an edge in common.)

42. [1] Mr. X brings some laundry from his house to a nearby river. After washing it in the river, he delivers it to Ms. Y who lives on the same side of the river.



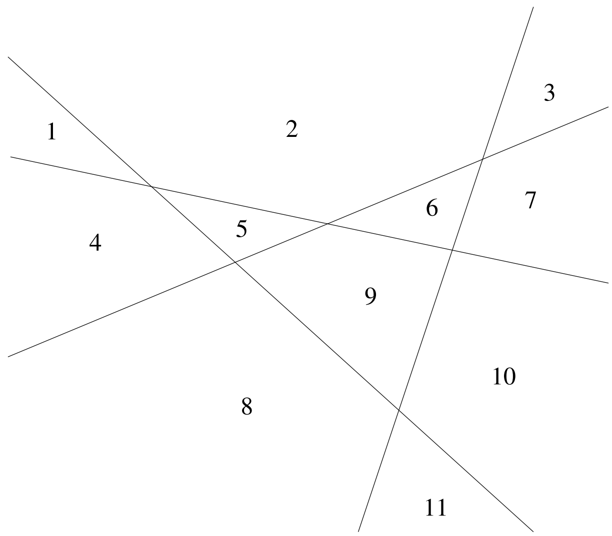
At what point on the river should Mr. X bring the laundry in order to travel the least possible distance? Try to do this problem without using calculus. (Assume of course that the river is a straight line.)

43. [2] An *antimagic square* is an $n \times n$ matrix whose entries are the distinct integers $1, 2, \ldots, n^2$ such that any set of n entries, no two in the same row or column, have the same sum of their elements. For instance,

$$\begin{bmatrix} 14 & 8 & 16 & 6 \\ 9 & 3 & 11 & 1 \\ 10 & 4 & 12 & 2 \\ 13 & 7 & 15 & 5 \end{bmatrix}$$

For what values of n do there exist $n \times n$ antimagic squares?

44. [1] Let $f(n)$ be the number of regions which are formed by n lines in the plane, where no two lines are parallel and no three meet in a point. E.g., $f(4) = 11$.



Find a simple formula for $f(n)$.

45. [2.5] Define a sequence $a_0, a_1, a_2, \ldots$ of integers as follows: $a_0 = 0$, and given $a_0, a_1, \ldots, a_n$, then a_{n+1} is the least integer greater than a_n such that no three distinct terms (not necessarily consecutive) of $a_0, a_1, \ldots, a_{n+1}$ are in arithmetic progression. (This means that for no $0 \le i < j < k \le n+1$ do we have $a_j - a_i = a_k - a_j$.) Find a simple rule for determining a_n. For instance, what is $a_{1000000}$? The sequence begins $0, 1, 3, 4, 9, 10, 12, \ldots$.

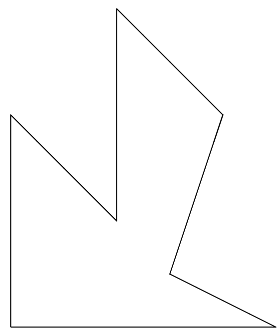
46. (a) [1] Let a, b, m, n be positive integers. Suppose that an $m \times n$ checkerboard can be tiled with $a \times b$ boards (in any orientation), i.e., the $a \times b$ boards can be placed on the $m \times n$ board to cover it completely, with no overlapping of the interiors of the $a \times b$ boards. Show that mn is divisible by ab.

    (b) [2.5] Assuming the condition of (a), show in fact that at least one of m and n is divisible by a. (Thus by symmetry, at least one of m and n is divisible by b.) For instance, a 6×30 board cannot be tiled with 4×3 boards.

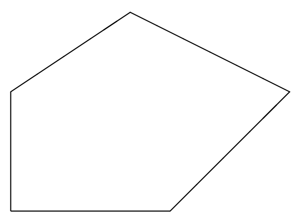
    (c) [2.5] Generalize (b) to any number of dimensions.

47. [2.5] Let R be a rectangle whose sides can have any positive real lengths. Show that if R can be tiled with finitely many rectangles all with at least one side of integer length, then R has at least one side of integer length.

48. [3] A *polygon* is a plane region enclosed by non-intersecting straight line segments, such as



A polygon P is *convex* if any straight line segment whose endpoints lie in P lies entirely in P. For instance, the above polygon is not convex, but



is convex. Can a convex polygon be dissected into non-convex quadrilaterals? (A quadrilateral is a four-sided polygon. The non-convex quadrilaterals in the above question may be of any size and shape, provided none are convex.) This problem was formulated and solved by a Berkeley undergraduate; none of the mathematics professors to whom he showed it were able to solve it.

49. (a) [3] Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Assume that f is a polynomial in each variable separately, i.e., for all $a \in \mathbb{R}$, the functions $f(a, x)$ and $f(x, a)$ are polynomials in x. Prove that $f(x, y)$ is a polynomial in x and y.

    (b) [2.5] Show that (a) is false if $\mathbb{R}$ is replaced by $\mathbb{Q}$ (the rational numbers).

50. [3.5] Does there exist a polynomial $f(x)$ with real coefficients such that $f(x)^2$ has fewer nonzero coefficients than $f(x)$?