Inequalities

Tomasz Ślusarczyk

Today, we will talk about two types of inequalities: "hard" and "soft". This distinction, albeit not widely recognized, is a great framework to think about inequalities and related problems, especially when transitioning from olympic problem-solving to mathematical research.

Before we begin, let us review some basic inequalities: AMGM (arithmetic mean – geometric mean), Cauchy-Schwarz inequality and Hölder's indequality. You might have seen these inequalities in the olympic setting in the discrete case, but the last two have also very useful continuous statements with integrals replacing sums.

AMGM	$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \ldots a_n}$ assumptions: $a_1, \ldots, a_n \geq 0$
Cauchy-Schwarz	$(a_1 b_1 + \cdots + a_n b_n)^2 \leq (a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2)$ assumptions: $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$ $\left(\int_R f(x)g(x)\,\mathrm{d}x\right)^2 \leq \left(\int_R f(x)^2\,\mathrm{d}x\right)\left(\int_R g(x)^2\,\mathrm{d}x\right)$ assumptions: $f, g : R \to \mathbb{R}$
Hölder	$a_1 b_1 + \cdots + a_n b_n \leq (a_1^p + \cdots + a_n^p)^{1/p}(b_1^q + \cdots + b_n^q)^{1/q}$ assumptions: $p, q > 0$, $\frac{1}{p} + \frac{1}{q} = 1$ $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}_{\geq 0}$ $\int_R f(x)g(x)\,\mathrm{d}x \leq \left(\int_R f(x)^p\,\mathrm{d}x\right)^{1/p}\left(\int_R g(x)^q\,\mathrm{d}x\right)^{1/q}$ assumptions: $p, q > 0$, $\frac{1}{p} + \frac{1}{q} = 1$ $f, g : R \to \mathbb{R}_{\geq 0}$

In the integral statements R is any region of integration (usually an interval on a real line, but it can be multi-dimensional object too).

Note that in all integral formulations we need to have some extra assumptions on f and g to ensure that integrals are well defined. The rule of thumb is

When all integrals in a known inequality are well defined, the inequality holds.

For any applications outside of integration theory, you'll essentially never encounter non-integrable functions. In particular, during Putnam integrability/measurability issues can be usually neglected.

Remark 1 (Equivalence between discrete and continuous statements). In general, discrete statement of most inequalities follow from continuous statements by taking f and g to be step functions (i.e. $f(x) = a_{\lfloor x \rfloor}$). On the other hand, continuous statements can also be derived from discrete statements for example by taking Riemann sums, but this requires further specification of the class of functions we are interested in (continuous, Riemann-integrable, etc.).

Cauchy-Schwarz and Hölder's inequality (especially in the integral form) are often used in the special case of $g \equiv 1$. To easily recognize this later on, we will write it explicitly here in the 1-dimensional case.

Cauchy-Schwarz

$$(a_1 + \cdots + a_n)^2 \leq n(a_1^2 + \cdots + a_n^2)$$

assumptions: $a_1, \ldots, a_n \in \mathbb{R}$

$$\left(\int_a^b f(x)\,\mathrm{d}x \right)^2 \leq (b-a) \int_R f(x)^2\,\mathrm{d}x$$

assumptions: $f, g : R \to \mathbb{R}$, $a < b$

Hölder

$$a_1 + \cdots + a_n \leq n^{1/q}(a_1^p + \cdots + a_n^p)^{1/p}$$

assumptions: $p, q > 0$, $\frac{1}{p} + \frac{1}{q} = 1$
$a_1, \ldots, a_n \in \mathbb{R}_{\geq 0}$

$$\int_a^b f(x)\,\mathrm{d}x \leq (b-a)^{1/q} \left(\int_a^b f(x)^p\,\mathrm{d}x \right)^{1/p}$$

assumptions: $p, q > 0$, $\frac{1}{p} + \frac{1}{q} = 1$
$f, g : R \to \mathbb{R}_{\geq 0}$, $a < b$

"Hard" vs "soft"

What are "hard" and "soft" inequalities? The distinction is entirely unrelated to difficulty, there are easy "hard" inequalities and difficult "soft" ones. The key idea is the *exactness*. Inequalities are soft if they hold "up to a constant", establish just convergence/divergence but without specific rate, we can choose some constants arbitrarily etc. On the other hand, hard inequalities are exact – all constants are exact, rates of growth/convergence need to be precise etc. For example, 99% of olympic inequalities are hard, while majority of measure theory, topology or functional analysis is built on soft inequalities. The distinction hopefully becomes more clear after inspecting the following examples.

Example 1 (Nesbitt's inequality)**.** *For any positive reals a, b, c the following inequality holds:*

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}\,.$$

In this inequality the choice of the constant $(3/2)$ is critical – should the left side be replaced by, say 5, the inequality would be trivially false, and should it be replaced by $1/2$ it would be trivially true. The only interesting case is when the constant term is equal to precisely $3/2$. Therefore, the inequality is hard.

Let us move on to the next example.

Example 2 (Stolz-Cesàro theorem)**.** *Let $a_1,\ a_2,\ \ldots$ and $b_1,\ b_2,\ \ldots$ be sequences of real numbers, such that $(b_n)_{n\geq 1}$ is strictly increasing and unbounded. Assume that the sequence $(a_{n+1}-a_n)/(b_{n+1}-b_n)$ converges and*

$$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = l\,.$$

Show that $\lim_{n\to\infty} a_n/b_n$ exists and is equal to l.

This theorem is much softer – we don't hae any specified rate of convergence and sequences (a_n) and (b_n) may behave arbitrarily. In general, many theorems from analysis are soft, but there are also some soft Putnam problems.

Example 3 (Putnam 2010, A6)**.** *Let $f : [0,\infty) \to \mathbb{R}$ be a strictly decreasing continuous function such that $\lim_{x\to\infty} f(x) = 0$. Prove that*

$$\int_0^\infty \frac{f(x)-f(x+1)}{f(x)}\,\mathrm{d}x$$

diverges.

This inequality is even softer. In the solution we just use very imprecise bounds and arbitrary constants. f may converge to 0 very fast, very slow or at some intermediate rate, but it doesn't matter for the solution (although it does change how fast the target integral diverges).

Problems

Problem 1 (Putnam 2014, B2)**.** *Suppose that f is a function on the interval $[1,3]$ such that $-1 \leq f(x) \leq 1$ for all x and $\int_1^3 f(x)\,\mathrm{d}x = 0$. How large can $\int_1^3 \frac{f(x)}{x}\,\mathrm{d}x$ be?*

Remark 2 (What is a function?)*.* This is the exact phrasing used at Putnam and, somewhat surprisingly, it does not give any assumptions about integrability of f. A more formal statement would specify that we are interested in measurable functions f. However, don't worry about measure-theoretic details – just assume any reasonable function is integrable and all integrals given in the problem are well-defined. In fact, you can even think of f as being continuous and the solution will not change much.

Remark 3 (What do we actually need to find?)*.* Naive interpretation of the problem statement would suggest that we're looking for a maximum of $\int_1^3 f(x)/x\,\mathrm{d}x$ when f satisfies given conditions. However, a priori we don't know if this maximum is attained by any function (and do we actually need to determine if it is attained as a part of the proof?). Most precise statement would ask for *supremum* (lowest upper bound) of possible values of the integral to avoid ambiguity.

Let us first discuss a motivation for a solution. For any f as in the problem statement define

$$I(f) = \int_1^3 \frac{f(x)}{x}\,\mathrm{d}x\,. \tag{1}$$

Consider some small ϵ, δ, $1 < a < b < 3$ and

$$f_1(x) = \begin{cases} f(x) + \epsilon & \text{when } a < x < a + \delta \\ f(x) - \epsilon & \text{when } b < x < b + \delta \\ f(x) & \text{otherwise} \end{cases} .$$

Think of f_1 as f with some mass shifted to the left.[1] Let $I(f) = \int_1^3 \frac{f(x)}{x}\,\mathrm{d}x$. Notice that (for very small ϵ, δ), we will have

$$I(f_1) \approx I(f) + \epsilon\delta/a - \epsilon\delta/b > I(f_1)\,.$$

Hence, intuitively, moving mass to the left increases I. Hence to maximize I we should perhaps move all the possible mass to the left. Since we are constrained by $-1 \le f(x) \le 1$ and $\int_1^3 f(x)\,\mathrm{d}x = 0$, the "best we can do" is

$$g(x) = \begin{cases} 1 & \text{if } 1 \le x \le 2 \\ \text{-1} & \text{if } 2 < x \le 3 \end{cases} . \tag{2}$$

The idea that such g should maximize the value of I motivates the solution below.

Solution. Let $I(f)$ and $g(x)$ be as in (1) and (2). We have

$$\begin{aligned} I(g) - I(f) &= \int_1^3 \frac{g(x) - f(x)}{x}\,\mathrm{d}x \\ &= \int_1^2 \frac{1 - f(x)}{x}\,\mathrm{d}x + \int_2^3 \frac{-1 - f(x)}{x}\,\mathrm{d}x \\ &\ge \int_1^2 \frac{1 - f(x)}{2}\,\mathrm{d}x + \int_2^3 \frac{-1 - f(x)}{2}\,\mathrm{d}x \\ &= \frac{1}{2} - \frac{1}{2}\int_1^2 f(x)\,\mathrm{d}x - \frac{1}{2} - \frac{1}{2}\int_2^3 f(x)\,\mathrm{d}x \\ &= -\frac{1}{2}\int_1^3 f(x)\,\mathrm{d}x = 0 \end{aligned}$$

where in the third line we used $1 - f(x) \ge 0$ (and bounded denominator by $x \le 2$) and $-1 - f(x) \le 0$ (and bounded denominator by $x \ge 2$). Therefore indeed $I(g) \ge I(f)$ for any f satisfying the problem conditions. Therefore the largest possible value of $I(f)$ is attained when $f = g$ and is equal to

$$\int_1^3 \frac{g(x)}{x}\,\mathrm{d}x = \int_1^2 \frac{1}{x}\,\mathrm{d}x - \int_2^3 \frac{1}{x}\,\mathrm{d}x = \ln\frac{4}{3}\,.$$

□

Was this problem hard or soft? We used definitely hard techniques – we sought exact maximum and the whole solution relied on the fact that we identified g as the function attaining exactly the maximum. It would not be sufficient to find a function which achieves half or some small fraction of maximum, so the problem was hard.

Problem 2 (Putnam 1988, B4)**.** *Assume that $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive real numbers. Show that $\sum_{n=1}^{\infty} a_n^{n/(n+1)}$ also converges.*

[1]Notice how this corresponds to the following operation: take a sequence $(a_1, \ldots, a_n)$ and change a_i to $a_i + \epsilon$ and a_j to $a_j + \epsilon$ for some $i < j$. You might recognize this technique as useful in some olympic problems, it's sometimes called the "shifting method".

Is this problem hard or soft? Well, arbitrary rate of convergence of $\sum a_n$ suggests that perhaps this problem is soft. Indeed, we will see that soft approach yields great results.

It would be very convenient, if we could bound $a_n^{n/(n+1)}$ by a constant multiple of a_n – say, $a_n^{n/(n+1)} \leq 10a_n$. Notice that

$$a_n^{n/(n+1)} \leq 10a_n \iff a_n \geq 10^{-n-1}\,. \tag{3}$$

This already looks very promising. If for all $n \geq 1$ we have $a_n >\geq 10^{-1/(n+1)}$, then we are done by (3):

$$\sum_{n=1}^{\infty} a_n^{\frac{n}{n+1}} \leq 10 \sum_{n=1}^{\infty} a_n < \infty\,.$$

On the other hand, what happens in the other extreme, when for all $n \geq 1$ we have $a_n < 10^{-n-1}$? Then the sequence a_n converges to 0 so fast, that series $a_n^{n/(n+1)}$ will converge. Indeed, $a_n^{n/(n+1)} < 10^{-n}$, so

$$\sum_{n=1}^{\infty} a_n^{\frac{n}{n+1}} < \sum_{n=1}^{\infty} 10^{-n} < \infty\,.$$

It may seem that we solved only two convenient cases, when either all a_n are large (i.e. larger than 10^{-n-1}) or small (smaller than 10^{-n-1}), but what happens in the intermediate cases? It turns out that it is not a problem at all.

Solution. Note that by (3) we have

$$a_n^{\frac{n}{n+1}} \leq 10a_n + 10^{-n} \tag{4}$$

since either $a_n > 10^{-n-1}$ and $a_n^{n/(n+1)} \leq 10a_n$ or $a_n < 10^{-n-1}$ and $a_n^{n/(n+1)} < 10^{-n}$. Therefore

$$\sum_{n=1}^{\infty} a_n^{\frac{n}{n+1}} \leq \sum_{n=1}^{\infty} \Big(10a_n + 10^{-n}\Big) \leq 10 \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} 10^{-n} < \infty$$

which finishes the proof. □

Remark 4. Note that we rearranged infinite sums free several times. We are allowed to do so, as all summands are nonnegative: if the infinite sum of nonnegative terms is convergent, rearranging terms will not change its convergence nor its limit (and if such sum is infinite, it will stay infinite).

Notice how soft mindset was applied to solve this problem. Constant 10 is completely arbitrary and in fact we can replace 10 by any $c > 1$. Moreover, notice how (4) is never sharp, we always need just one summand. This step may seem trivial, but I want to highlight it – in hard inequalities (in particular in mathematical olympiads) such steps basically nveer work since we look for exact bounds and this one by design can never be tight. However, since we work in the soft setting, we are fine with loosing tightness if it allows us to easily combine two extreme cases into general proof.

Problem 3 (Blakley-Roy inequality)**.** *Show that for any* $f : [0,1]^2 \to \mathbb{R}_{\geq 0}$

$$\int_{[0,1]^4} f(x,y)f(z,y)f(z,w)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w \geq \left(\int_{[0,1]^2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y\right)^3\,.$$

Remark 5. This theorem was published independently at least 3 times between 1959 and 1965, but gained most recognition after 1965 presentation by Blakley and Roy.

Proof. Instead of writing $\int_0^1 \int_0^1 \int_0^1 \dots \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ I will write $\int_z \int_y \int_x \dots$ to easily identify which integral corresponds to which variable. In all cases all integrals are taken over $[0,1]$ in each variable.

Let us define $g(x) = \int_y f(x,y)$ and let

$$L = \int_x \int_y \int_z \int_z f(x,y)f(z,y)f(z,w)$$

be the left side of the target inequality. We can rearrange integrals to integrate w out:

$$\begin{aligned} L &= \int_x \int_y \int_z \int_z f(x,y)f(z,y)f(z,w) \\ &= \int_x \int_y \int_z \left(f(x,y)f(z,y) \int_w f(z,w) \right) \\ &= \int_{x,y,z} f(x,y)f(z,y)g(z)\,. \end{aligned}$$

By swapping x and z we conclude that also

$$L = \int_{x,y,z} f(x,y)f(z,y)g(z) = \int_{x,y,z} f(x,y)f(z,y)g(x)\,.$$

Therefore by taking average of these two expressions

$$\begin{aligned} L &= \int_{x,y,z} f(x,y)f(z,y)\frac{g(z)+g(x)}{2} \\ &\geq \int_{x,y,z} f(x,y)f(z,y)\sqrt{g(z)}\sqrt{g(x)} \\ &= \int_y \left(\int_x \int_z f(x,y)\sqrt{g(x)}f(z,y)\sqrt{g(z)} \right) \qquad (5) \\ & \qquad (6) \end{aligned}$$

Now note that for any fixed y we have

$$\begin{aligned} &\int_x \int_z f(x,y)\sqrt{g(x)}f(z,y)\sqrt{g(z)} \\ &\quad = \int_x \left(f(x,y)\sqrt{g(x)} \int_z f(z,y)\sqrt{g(z)} \right) \\ &\quad = \left(\int_x f(x,y)\sqrt{g(x)} \right) \left(\int_z f(z,y)g(z) \right) \\ &\quad = \left(\int_x f(x,y)\sqrt{g(x)} \right)^2 . \qquad (7) \end{aligned}$$

Notice the analogy with contracting double sum to a square:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j = \left(\sum_{i=1} a_i \right)^2$$

which is evident by observing what products appear when we expand the square.

Now by integrating (7) we obtain

$$\int_y \left(\int_x \int_z f(x,y)\sqrt{g(x)}f(z,y)\sqrt{g(z)} \right) = \int_y \left(\int_x f(x,y)\sqrt{g(x)} \right)^2$$

so we can plug this into (5) to write

$$\begin{aligned} L &\geq \int_y \left(\int_x \int_z f(x,y)\sqrt{g(x)} f(z,y) \sqrt{g(z)} \right) \\ &= \int_y \left(\int_x f(x,y)\sqrt{g(x)} \right)^2 . \end{aligned}$$

Now we can use Cauchy-Schwarz (in the integral form, of course), in the case when of the functions is identically equal to 1. Since we integrate over $x \in [0,1]$ which has length 1, we continue

$$\begin{aligned} &\geq \left(\int_y \int_x f(x,y)\sqrt{g(x)} \right)^2 \\ &= \left(\int_x \int_y f(x,y)\sqrt{g(x)} \right)^2 \\ &= \left(\int_x g(x) \int_y f(x,y) \right)^2 \\ &= \left(\int_x g(x)^{3/2} \right)^2 \end{aligned} \tag{8}$$

Finally, we can finish by using Hölder's inequality (also in the case when one of the functions is identically equal to 1) for $p = 3/2$ to obtain

$$\left(\int_x g(x)^{3/2} \right)^{2/3} \geq \int_x g(x)$$

so that

$$L \geq \left(\int_x g(x)^{3/2} \right)^2 \geq \left(\int_x g(x) \right)^3 = \left(\int_x \int_y f(x,y) \right)^3$$

which finishes the proof. □

Remark 6. We changed order of integration numerous times throughout the solution. It is possible since all of our integrands are nonnegative, so for example Fubini theorem applies. Similarly, sums of nonnegative terms can be also rearranged quite freely.

This inequality was hard. We used precise bounds and there was no place in the proof were we could replace something with an arbitrary constant.

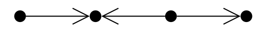
This type of integral inequality may seem very weird to consider. I'm not sure what the original motivation was, but some more or less standard (at least standard nowadays) techniques of graph theory show that this inequality is essentially equivalent to the following theorem (a special case of Sidorenko's conjecture).

Theorem 1. *Let G be any oriented graph, that is $T = (V, E)$, where V is the set of vertices and $E \subset V^2$ is the set of edges, which allows edges in both directions and loops, but no multiple edges between two vertices with the same orientation. Let $|V| = n$ and $|E| = m$. Then the number of quadruples of vertices (x, y, z, w) such that $xy, zy, zw \in E$ is at least m^2/n^2.*

Corollary 1. *In any tournament on n verticed there are at least*

$$\Big(1 - o(1)\Big)\frac{n^4}{8}$$

paths of the form



where $o(1)$ *is a function of* n *converging to 0 when* $n \to \infty$.

If you want to learn more about relations between graph theory and inequalities between integrals, there is a class at MIT taught by prof. Zhao covering this topic in great detail: 18.225 Graph Theory and Additive Combinatorics.