PROBLEMS ON GENERATING FUNCTIONS

1. Let $a_n = (n^2+1)3^n$. Compute $y = \sum_{n\geq 0} a_n x^n$.

2. Compute
$$y = x + \frac{2}{3}x^3 + \frac{2}{3}\frac{4}{5}x^5 + \frac{2}{3}\frac{4}{5}\frac{6}{7}x^7 + \cdots.$$

3. Given $a_0 = 2$, $a_1 = 3$, and
$$(n+1)(n+2)a_{n+2} - 3(n+1)a_{n+1} + 2a_n = 0,$$
for $n \geq 0$, compute $y = \sum_{n\geq 0} a_n x^n$.

4. Given $a_0 = 1$ and $a_{n+1} = (n+1)a_n - \binom{n}{2}a_{n-2}$ for $n \geq 0$, compute $y = \sum_{n\geq 0} a_n \frac{x^n}{n!}$.

5. Let k be a positive integer and let $m = 6k-1$. Let
$$S(m) = \sum_{j=1}^{2k-1} (-1)^{j+1}\binom{m}{3j-1}.$$
For example with $k=3$,
$$S(17) = \binom{17}{2} - \binom{17}{5} + \binom{17}{8} - \binom{17}{11} + \binom{17}{14}.$$
Prove that $S(m)$ is never zero.

6. For nonnegative integers n and k, define $Q(n,k)$ to be the coefficient of x^k in the expansion of $(1+x+x^2+x^3)^n$. Prove that
$$Q(n,k) = \sum_{j=0}^{n} \binom{n}{j}\binom{n}{k-2j}.$$

7. Let $a_{m,n}$ denote the coefficient of x^n in the expansion of $(1+x+x^2)^m$. Prove that for all $k \geq 0$,
$$0 \leq \sum_{i=0}^{\lfloor 2k/3 \rfloor} (-1)^i a_{k-i,i} \leq 1.$$

8. Consider the power series expansion
$$\frac{1}{1-2x-x^2} = \sum_{n=0}^{\infty} a_n x^n.$$
Prove that, for each integer $n \geq 0$, there is an integer m such that $a_n^2 + a_{n+1}^2 = a_m$.

9. Let $A = \{(x,y) : 0 \leq x, y \leq 1\}$. For $(x,y) \in A$, let
$$S(x,y) = \sum_{\frac{1}{2} \leq \frac{m}{n} \leq 2} x^m y^n,$$
where the sum ranges over all pairs (m,n) of positive integers satisfying the indicated inequalities. Evaluate
$$\lim_{(x,y)\to(1,1),\ (x,y)\in A} (1-xy^2)(1-x^2y)S(x,y).$$

10. For a set S of nonnegative integers, let $r_S(n)$ denote the number of ordered pairs (s_1, s_2) such that $s_1 \in S$, $s_2 \in S$, $s_1 \neq s_2$, and $s_1 + s_2 = n$. Is it possible to partition the nonnegative integers into two sets A and B in such a way that $r_A(n) = r_B(n)$ for all n?

11. For positive integers m and n, let $f(m,n)$ denote the number of n-tuples $(x_1, x_2, \ldots, x_n)$ of integers such that $|x_1| + |x_2| + \cdots + |x_n| \leq m$. Show that $f(m,n) = f(n,m)$.

12. Let S_n denote the set of all permutations of the numbers $1, 2, \ldots, n$. For $\pi \in S_n$, let $\sigma(\pi) = 1$ if π is an even permutation and $\sigma(\pi) = -1$ if π is an odd permutation. Also, let $\nu(\pi)$ denote the number of fixed points of π. Find

$$\sum_{\pi \in S_n} \frac{\sigma(\pi)}{\nu(\pi) + 1}.$$

13. Let S be the set of sequences of length 2018 whose terms are in the set $\{1, 2, 3, 4, 5, 6, 10\}$ and sum to 3860. Prove that the cardinality of S is at most

$$2^{3860} \left(\frac{2018}{2048} \right)^{2018}.$$

14. Suppose that $\mathbb{Z}$ is written as a disjoint union of $n < \infty$ arithmetic progressions, with differences $d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$. Show that $d_1 = d_2$.

15. Solve the following equation for the power series $F(x,y) = \sum_{m,n \geq 0} a_{mn} x^m y^n$, where $a_{mn} \in \mathbb{R}$:

$$(xy^2 + x - y)F(x,y) = xF(x,0) - y.$$

The point is to make sure that your solution has a power series expansion at $(0,0)$.

16. Find a simple description of the coefficients $a_n \in \mathbb{Z}$ of the power series $F(x) = x + a_2 x^2 + a_3 x^3 + \cdots$ satisfying the functional equation

$$F(x) = (1+x)F(x^2) + \frac{x}{1-x^2}.$$

17. Consider the power series $y = \sum_{n=0}^{\infty} \binom{3n}{n} x^n = 1 + 3x + 15x^2 + \cdots$. Show that

$$(27x - 4)y^3 + 3y + 1 = 0.$$

18. Find the unique power series $y = 1 + \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \frac{1}{30240}x^6 + \cdots$ such that for all $n \geq 0$, the coefficient of x^n in y^{n+1} is equal to 1.

19. Find the unique power series $y = 1 + x - \frac{1}{2}x^2 + \frac{2}{3}x^3 + \cdots$ such that the constant term is 1, the coefficient of x is 1, and for all $n \geq 2$ the coefficient of x^n in y^n is 0.

20. Let $f(m,0) = f(0,n) = 1$ and $f(m,n) = f(m-1,n) + f(m,n-1) + f(m-1,n-1)$ for $m, n > 0$. Show that

$$\sum_{n=0}^{\infty} f(n,n)x^n = \frac{1}{\sqrt{1 - 6x + x^2}}.$$