PROBLEMS ON COMBINATORIAL CONFIGURATIONS

These problems aim to capture the flavors of combinatorial questions which have been common in recent years on the Putnam. Difficulty is very roughly increasing, with the first several being more exercise-like.

1. Can a 2018×2018 grid be tiled with rotations and reflections of the L-tetromino?

2. Each edge of a complete graph on $2k$ vertices is removed with probability $\frac{1}{2}$. Prove that with probability greater than $\frac{1}{4^k}$, the maximum degree of the remaining graph is at most $k-1$.

3. Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers,
$$n = a_1 + a_2 + \ldots + a_k,$$
with k an arbitrary positive integer and $a_1 \leq a_2 \leq \ldots \leq a_k \leq a_1 + 1$?

4. How many n-digit numbers whose digits are in the set $\{2, 3, 7, 9\}$ are divisible by 3?

5. A 6×6 grid is tiled by dominoes. Prove that there exists some line which cuts the board into two nonempty parts without cutting through any domino.

6. A collection of 2020 sets, each of size 42, has the property that any two of the sets have exactly one common element. Must all of the sets have a common element?

7. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

8. Given a positive integer n, what is the largest k such that the numbers $1, 2, \ldots, n$ can be put into k boxes such that the sum of the numbers in each box is the same?

9. A class with $2N$ students took a quiz, where each student received a integer score between 0 and 10 inclusive. At least one student received each possible score, and the average score was exactly 7.4. Show that the class can be divided into two groups with equal number of students and the same average score.

10. Callie and Marie play a game in which they take turns filling entries of an initially empty 2008×2008 array. Callie plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Callie wins if the determinant of the resulting matrix is nonzero; Marie wins if it is zero. Which player has a winning strategy?

11. Consider a $(2m-1) \times (2n-1)$ rectangular region, where m, n are integers such that $m, n \geq 4$. This region is to be tiled without overlap using rotations and reflections of the bent triomino and the S-tetromino. What is the fewest number of these tiles which can be used?

12. Let B be a set of more than $2^{n+1}/n$ distinct points with coordinates of the form $\{\pm 1, \ldots, \pm 1\}$ in n-dimensional space with $n \geq 3$. Show that there are three distinct points in B which are the vertices of an equilateral triangle.

13. A set of n points is given in the plane such that the distance between any two of them is greater than 1. Prove that one can choose at least $\frac{n}{7}$ of these points such that the distance between any two of them is greater than $\sqrt{3}$.

14. A round-robin tournament of $2n$ teams lasted for $2n-1$ days, as follows. On each day, every team played one game against another team, with one team winning and one team losing in each of the n games. Over the course of the tournament, each team played every other team exactly once. Can one necessarily choose one winning team from each day without choosing any team more than once?

15. Callie and Marie play a game with r red and g green stones. At every step, one can remove an amount k of stones of either color, where k divides the number of stones remaining of the other color. She who moves last wins, and Callie starts. For which (r, g) does Callie win?

16. Consider a polyhedron with at least five faces such that exactly three edges emerge from each of its vertices, and no two edges share more than one face. Two players play the following game:

    Each player, in turn, signs his or her name on a previously unsigned face. The winner is the player who first succeeds in signing three faces that share a common vertex.

    Show that the player who signs first will always win by playing as well as possible.

17. Two hundred students participated in a mathematical contest. They had six problems to solve. It is known that each problem was correctly solved by at least 120 participants. Prove that there must be two participants such that every problem was solved by at least one of these two students.

18. A city is a point on the plane. Suppose there are $n \geq 2$ cities. Suppose that for each city X, there is another city $N(X)$ that is strictly closer to X than all the other cities. The government builds a road connecting each city X and its $N(X)$; no other roads have been built. Suppose we know that, starting from any city, we can reach any other city through a series of roads.

    We call a city Y *suburban* if it is $N(X)$ for some city X. Show that there are at least $(n-2)/4$ suburban cities.

19. Let $n \geq 2$ be an integer and T_n be the number of nonempty subsets S of $\{1, 2, 3, \ldots, n\}$ with the property that the average of the elements in S is an integer. Prove that $T_n - n$ is always even.

20. Let $\mathbb{Z}^n$ be the integer lattice in $\mathbb{R}^n$. Two points in $\mathbb{Z}^n$ are called *neighbors* if they differ by exactly 1 in one coordinate and are equal in all other coordinates. For which

integers $n \geq 1$ does there exist a set of points $S \subset \mathbb{Z}^n$ satisfying the following two conditions?

(a) If p is in S, then none of the neighbors of p is in S.

(b) If $p \in \mathbb{Z}^n$ is not in S, then exactly one of the neighbors of p is in S.

21. Prove that it is not possible to color the squares of a 11×11 grid using three colors such that no four squares whose centers form the vertices of a rectangle with sides parallel to the sides of the grid, have the same color.

22. A circle is divided by $2n$ evenly spaced points into $2n$ equal arcs. Half of these arcs are colored blue and half are colored red. The blue arcs are numbered from 1 to n in the counterclockwise direction, starting from an arbitrary blue arc, and red arcs are numbered from 1 to n in the clockwise direction, starting from an arbitrary red arc. Prove that one can find n consecutive arcs, each of which has a different label.

23. Let n and k be positive integers. Cathy is playing the following game. There are n marbles and k boxes, with the marbles labelled 1 to n. Initially, all marbles are placed inside one box. Each turn, Cathy chooses a box and then moves the marbles with the smallest label, say i, to either any empty box or the box containing marble $i+1$. Cathy wins if at any point there is a box containing only marble n. Determine all pairs of integers (n, k) such that Cathy can win this game.

24. Callie and Marie play a game in which they take turns choosing an element from the group of invertible $n \times n$ matrices with entries in the field $\mathbb{Z}/p\mathbb{Z}$ of integers modulo p, where n is fixed positive integer and p is a fixed prime number, The rules of the game are:

    - A player cannot choose an element that has been chosen by either player on any previous turn.
    - A player can only choose an element that commutes with all previously chosen elements.
    - A player who cannot choose an element on his/her turn loses the game.

    Callie takes the first turn. For which (n, p) does Callie have a winning strategy?

25. Suppose a finite number of integer arithmetic progressions are given, such that every positive integer belongs to exactly one of them. Prove that two of these progressions have the same difference.

26. Suppose you are given a binary string which is not entirely zeroes. Prove that you may insert + between some of the digits so that the resulting summation, when carried out in base 2, yields a power of 2. Example: for 101111 the split $1 + 0 + 1111$ is possible.

27. An alphabet consists of three letters. Some of the words of length 2 or more are prohibited, and all of the prohibited sequences have different lengths. A *word* is a sequence of letters of any length. A *correct* word does not contain any prohibited sequence. Prove that there are correct words of any length.

28. There are 2007 senators in a senate. Each senator has enemies within the senate. Prove that there is a non-empty subset K of senators such that, for every senator in the senate, the number of enemies of that senator in the set K is an even number.

29. Call a subset S of $\{1, 2, \ldots, n\}$ mediocre if it has the following property: Whenever a, b are elements of S whose average is an integer, that average is also an element of S. Let $A(n)$ be the number of mediocre subsets of $\{1, 2, \ldots, n\}$. Find all positive integers n such that $A(n+2) + A(n) = 2A(n+1) + 1$.

30. Let $S_1, S_2, \ldots, S_m$ be distinct subsets of $\{1, 2, \ldots, n\}$ such that $|S_i \cap S_j| = 1$ for all $i \neq j$. Prove that $m \leq n$.

31. For a set S of nonnegative integers, let $r_S(n)$ denote the number of ordered pairs (s_1, s_2) such that $s_1 \in S, s_2 \in S, s_1 \neq s_2, s_1 + s_2 = n$. Is it possible to partition the nonnegative integers into two sets A, B in such a way that $r_A(n) = r_B(n)$ for all n?

32. The 30 edges of a regular icosahedron are distinguished by labeling them $1, 2, \ldots, 30$. How many different ways are there to paint each edge red, white, or blue such that each of the 20 triangular faces of the icosahedron has two edges of the same color and a third edge of a different color?

33. There are 2010 boxes labeled $B_1, B_2, \ldots, B_{2010}$ and $2010n$ balls have been distributed among them, for some positive integer n. You may redistribute the balls by a sequence of moves, each of which consists of choosing an i and moving exactly i balls from box B_i into any one other box. For which values of n is it possible to reach the distribution with exactly n balls in each box, regardless of the initial distribution of balls?