Problem 1. Prove that there is a constant C such that, if $p(x)$ is a polynomial of degree 1999, then

$$|p(0)| \leq C \int_{-1}^{1} |p(x)| \, dx.$$

Solution 1:
Consider $p(x) = a_{1999}x^{1999} + a_{1998}x^{1998} + \cdots + a_0$, where not all the coefficients are zero. We wish to prove a bound of the form

$$f(a_0, \ldots, a_{1999}) = \frac{|a_0|}{\int_{-1}^{1} |a_{1999}x^{1999} + \cdots + a_0| \, dx} \leq C.$$

Note that as a function $f : \mathbb{R}^{2000} \backslash \{\vec{0}\} \to \mathbb{R}$, this is continuous. To check this, we note that $\vec{a} \mapsto |a_{1999}x^{1999} + \cdots + a_0|$ is continuous for each $x \in [-1, 1]$, and in fact $\vec{a}$ and $\vec{b}$ map to values differing by at most $\sum_{i=0}^{1999} |a_i - b_i|$. Thus the integral in question differs by at most twice this, which goes to zero as $\vec{b} \to \vec{a}$. Additionally, this map is nonzero when $\vec{a}$ is nonzero, since the integral of a nonzero continuous function is nonzero. Thus inverting it remains well-defined and hence continuous.

We want a uniform upper bound. Notice that f is scale-invariant, and thus the set of values attained by f is the same as the set of value obtained by f on $S^{1999} \subseteq \mathbb{R}^{2000}$. Now, the sphere is compact, and the image of a compact set is compact, so $f(\mathbb{R}^{2000} \backslash \{\vec{0}\}) = f(S^{1999}) \subseteq \mathbb{R}$ is compact. Compact subsets of $\mathbb{R}$ are bounded, by the Heine-Borel theorem, and if we let C be an upper bound on the magnitudes of elements in the compact set $f(S^{1999})$, then the above inequality clearly holds. We are done. $\square$

Note: by attempting to prove the stronger inequality $\sup_{x \in [-1,1]} |p(x)| \leq C \int_{-1}^{1} |p(x)| \, dx$ for some C, we can actually find that the problem is a direct consequence of the equivalence of all norms on $\mathbb{R}^n$.

Solution 2:
These compactness-style arguments tell us a number exists, but don't tell us which number. It is also worth seeing a more constructive argument with explicit bounding. This approach has its merits. Basically, we want a lower bound on the value of $|p(x)|$ over some interval of some fixed size. How do we lower-bound the magnitude of a polynomial? See how far the value is from the roots!

We are given some polynomial $p(x)$, which factors as $p(x) = \prod_{i=1}^{1999} (x - r_i)$, where r_i

are in the complex plane. Now

$$\frac{|p(x)|}{|p(0)|} = \prod_{i=1}^{1999} \frac{|x - r_i|}{|r_i|}. \tag{**}$$

Here's the idea: Fix some $\epsilon > 0$ (we will choose it later). Then we look at all x which are distance at least than ϵ from each of $r_1, \ldots, r_{1999}$. For technical reasons we only look at x in the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$. Then

$$\frac{|x - r_i|}{|r_i|} \geq \epsilon$$

if $|r_i| \leq 1$ and

$$\frac{|x - r_i|}{|r_i|} = \left|1 - \frac{x}{r_i}\right| \geq 1 - \left|\frac{x}{r_i}\right| \geq \frac{1}{2}$$

if $|r_i| \geq 1$, using that $|x| \leq \frac{1}{2}$. Thus, returning to (**), we have

$$\frac{|p(x)|}{|p(0)|} \geq \prod_{i=1}^{1999} \min\left(\epsilon, \frac{1}{2}\right) = \min\left(\epsilon, \frac{1}{2}\right)^{1999}$$

for these x. Finally, we choose ϵ so that there are "a lot" of x which satisfy the imposed conditions: $|x - r_i| \geq \epsilon$ for $i \in [1999]$ and $|x| \leq \frac{1}{2}$. Well, choose $\epsilon = \frac{1}{4000}$. Each "bad disk" of the form $|z - r_i| \leq \epsilon$ can cover at most a length 2ϵ interval of the real numbers, for a total of 3998ϵ (in length). The interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$ has length 1. Thus at least a length of $1 - 3998\epsilon = \frac{1}{2000}$ is uncovered (there is technically something nontrivial in this statement: alternatively, there are at most 2000 intervals in the complement, and one of them has length at least $\frac{1}{2000^2}$), so the integral $\int_{-1}^{1} \frac{|p(x)|}{|p(0)|}\, dx$ is at least

$$\frac{1}{2000} \cdot \left(\frac{1}{4000}\right)^{1999}.$$

In general, for degree n polynomials, this gives $C(n) = \exp(O(n \log n))$ or something. Can you do better? □

Problem 2. Let k be an integer greater than 1. Suppose $a_0 > 0$, and define

$$a_{n+1} = a_n + \frac{1}{\sqrt[k]{a_n}}$$

for $n > 0$. Evaluate

$$\lim_{n\to\infty} \frac{a_n^{k+1}}{n^k}.$$

Solution 1:
We'll first give a solution that uses some important bounding ideas, but there is a second more natural solution that we will get to. First, what should the answer be? The question implies that $a_n = c_k n^{\frac{k}{k+1}}$ is a good model of the growth. Then

$$c_k(n+1)^{\frac{k}{k+1}} - c_k n^{\frac{k}{k+1}} \approx \frac{1}{c_k^{\frac{1}{k}} n^{\frac{1}{k+1}}}.$$

The left difference looks like a derivative so is around $\frac{c_k \cdot k}{k+1} n^{-\frac{1}{k+1}}$, so we would expect $c_k^{\frac{k+1}{k}} = \frac{k+1}{k}$, for a final answer of $\left(\frac{k+1}{k}\right)^k$. Of course, this all assumes that the limit even exists. Okay, so maybe the natural thing to look at is $a_n^{\frac{k+1}{k}}$, which we expect to be around $\frac{k+1}{k}n$ (this expression has no fractional exponents in sight). Well, if we can show that $a_{n+1}^{\frac{k+1}{k}} - a_n^{\frac{k+1}{k}}$ is "near" $\frac{k+1}{k}$, then we should be morally done.
Well,

$$\left|a_{n+1}^{\frac{k+1}{k}} - a_n^{\frac{k+1}{k}} - \frac{k+1}{k}\right| = \left|a_n^{\frac{k+1}{k}}\right| \left|\left(1 + \frac{1}{a_n^{\frac{k+1}{k}}}\right)^{\frac{k+1}{k}} - 1 - \left(\frac{k+1}{k}\right) a_n^{-\frac{k+1}{k}}\right|$$

looks very promising due to the Taylor approximation $(1+x)^p \approx 1 + px + \frac{p(p-1)}{2}x^2 + O(x^3)$. In fact, for each p there is ϵ_p so that $x < \epsilon_p$ implies $(1+x)^p \geq 1 + px$ (true at $x = 0$ and take derivatives) and $(1+x)^p \leq 1 + px + \left(\frac{p(p-1)}{2} + 1\right) x^2$, true by looking at Taylor series. Now applying this for the constant $p = \frac{k+1}{k}$ we find an estimate

$$\begin{aligned}\left|a_n^{\frac{k+1}{k}}\right| \left|\left(1 + \frac{1}{a_n^{\frac{k+1}{k}}}\right)^{\frac{k+1}{k}} - 1 - \left(\frac{k+1}{k}\right) a_n^{-\frac{k+1}{k}}\right| &\leq \left|a_n^{\frac{k+1}{k}}\right| \left|\left(\frac{\frac{k+1}{k} \cdot \frac{1}{k}}{2} + 1\right) a_n^{-\frac{k+1}{k} \cdot 2}\right| \\ &\leq C_k a_n^{-\frac{k+1}{k}}.\end{aligned}$$

This is really good. Now let's justify that $a_n \to \infty$ as $n \to \infty$. Indeed, notice that $a_0 \le a_1 \le \cdots$ so that $\frac{1}{a_n^{\frac{1}{k+1}}} \le \frac{1}{a_0^{\frac{1}{k+1}}} = D$. Then $a_n \le a_0 + Dn$, so $a_{n+1} \ge a_n + \frac{1}{(a_0+Dn)^{\frac{1}{k+1}}}$. Since $\frac{1}{k+1} < 1$, hashing out this series we see that it diverges, so $a_n \to \infty$. We can also just note that a_n is increasing so if not unbounded it tends toward a limit, implying that $a_{n+1} - a_n = \frac{1}{a_n^{\frac{1}{k+1}}}$ eventually approaches 0. But this makes no sense, since this forces a_n to get arbitrarily large.

Okay, so formally, we have proved that

$$\lim_{n\to\infty}\left(a_{n+1}^{\frac{k+1}{k}} - a_n^{\frac{k+1}{k}}\right) = \frac{k+1}{k}.$$

I claim that this means

$$\lim_{n\to\infty}\frac{a_n^{\frac{k+1}{k}}}{n} = \frac{k+1}{k},$$

which is what we want.

In fact, more is true:

Lemma. (Cesàro's Lemma) If $\lim_{n\to\infty} a_n = c$ then $\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n a_i = c$.

Proof:

Fix $\epsilon > 0$. Then for $n > N$ we have $|a_n - c| < \epsilon$, hence $\frac{1}{n}\sum_{i=1}^n a_i \ge \frac{1}{n}(a_1 + \cdots + a_N + (c-\epsilon)(n-N))$ and similarly it is less than $\frac{1}{n}(a_1 + \cdots + a_N + (c+\epsilon)(n-N))$. Thus we see that the ratio eventually is in the range $(c-2\epsilon, c+2\epsilon)$ for n sufficiently large. That is, the limit is c. $\square$

We are done! $\square$

Solution 2: Since we have time, let's look at the second approach. First, fix k and assume without loss of generality that $a_0 \ge k^{\frac{k}{k+1}}$ (we'll see why we need this in a bit). Why can we assume this? Well, we can always shift the starting index of the sequence a little bit, and we know that $a_n \to \infty$ through an elementary argument (see above).

Now the idea is to look at $\frac{a_{n+1}-a_n}{(n+1)-n} = \frac{1}{\sqrt[k]{a_n}}$ as a discretization of some differential equation (this is basically Euler's method, applied to infinity). In particular, the equation is $\frac{dy}{dx} = \frac{1}{\sqrt[k]{y}}$ with boundary condition $f(0) = a_0 > 0$. It is well-known how to solve these: we see $y^{\frac{1}{k}}\frac{dy}{dx} = 1$, and integrating dx gives $\frac{k}{k+1}y^{\frac{k+1}{k}} = x + a_0^{\frac{k+1}{k}}$. Then $y(x) = \left(\left(\frac{k+1}{k}\right)(x+d)\right)^{\frac{k}{k+1}}$ for $d = a_0^{\frac{k+1}{k}}$.

Two things to notice: $y(0) = a_0$ and $y'(0)$ is decreasing, i.e., y is concave. Thus it is not hard to check that $y(x+1) - y(x) = \int_x^{x+1} y'(t)dt \leq y'(x)$, so that $y(x+1) \leq y(x) + \frac{1}{\sqrt[k]{y(x)}}$.

Thus the sequence $b_n = y(n)$ satisfies $b_{n+1} \leq b_n + \frac{1}{\sqrt[k]{b_n}}$ for $n \geq 0$ and $b_0 = a_0$. It's clear that $b_0 \leq a_0$, and we can induct to show that $b_n \leq a_n$ for all $n \geq 0$: since $b_n \leq a_n$ we have $b_{n+1} \leq b_n + \frac{1}{\sqrt[k]{b_n}} \leq a_n + \frac{1}{\sqrt[k]{a_n}} = a_{n+1}$. We know the middle inequality since $f(x) = x + \frac{1}{x^{\frac{1}{k}}}$ is increasing for $x > k^{\frac{k+1}{k}}$ by taking derivatives (this is where the strange constant comes in!).

Now $b_n = y(n) = \left(\frac{k+1}{k}\right)^{\frac{k}{k+1}} (n+d)^{\frac{k}{k+1}}$, so $a_n \geq (n+d)^{\frac{k}{k+1}}$. This is a pretty explicit bound, and you can also prove it directly by induction (again only if we start with $a_0 \geq k^{\frac{k}{k+1}}$), but this tells you what the correct lower bound to induct on is, and simplifies the inequalities involved in the induction step by allowing us to express the difference $b_{n+1} - b_n$ as an integral that then has nice bounding properties.

Anyways, this expression gives the right constant that we want: we just need an upper bound. How do we do this? Find a way to invert the inequality! We have

$$a_{n+1} = a_n + \frac{1}{\sqrt[k]{a_n}} \leq a_n + \frac{1}{\sqrt[k]{b_n}},$$

so expanding gives

$$a_n \leq a_0 + \sum_{i=0}^{n-1} \frac{1}{\sqrt[k]{b_n}}.$$

If you think about this a little, it's clear that this gives the "right expression" for an upper bound. More precisely, this is

$$a_n \leq a_0 + \sum_{i=0}^{n-1} \frac{1}{\left(\frac{k+1}{k}\right)^{\frac{1}{k+1}} (n+d)^{\frac{1}{k+1}}} \leq a_0 + s + \frac{1}{\left(\frac{k+1}{k}\right)^{\frac{1}{k+1}}} \sum_{i=1}^{n-1} \frac{1}{i^{\frac{1}{k+1}}}$$

where s is some inconsequential constant like $\frac{1}{\left(\frac{k+1}{k}\right)^{\frac{1}{k+1}} d^{\frac{1}{k+1}}}$. Finally, remember some tricks from Evan's lecture: this series looks like a Riemann sum for the integral $\int_1^n \frac{1}{x^{\frac{1}{k+1}}}\, dx$, and it is not hard to upper bound the right by

$$a_n \leq a_0 + s + \frac{1}{\left(\frac{k+1}{k}\right)^{\frac{1}{k+1}}} \left(1 + \int_1^n x^{-\frac{1}{k+1}}\, dx\right).$$

Finally,

$$a_n \le a_0 + s + \frac{1}{\left(\frac{k+1}{k}\right)^{\frac{1}{k+1}}} \cdot \left(\frac{k+1}{k} n^{\frac{k}{k+1}} - \frac{1}{k}\right) = a_0 + s' + \left(\frac{k+1}{k}\right)^{\frac{k}{k+1}} n^{\frac{k}{k+1}}.$$

The upper and lower bounds are of the same order and trivially give

$$\lim_{n\to\infty} \frac{a_n^{k+1}}{n^k} = \left(\frac{k+1}{k}\right)^k,$$

as claimed. □