PROBLEMS ON LINEAR ALGEBRA

1 Basic Linear Algebra

1. Let M_n be the $(2n+1)\times(2n+1)$ for which

$$(M_n)_{ij} = \begin{cases} 0, & i = j \\ 1, & i - j \equiv 1, \ldots, n \pmod{2n+1} \\ -1, & i - j \equiv n+1, \ldots, 2n \pmod{2n+1}. \end{cases}$$

Find the rank of M_n.

2. Let a_{ij} $(i, j = 1, 2, 3)$ be real numbers such that $a_{ij} > 0$ for $i = j$, and $a_{ij} < 0$ for $i \neq j$. Prove that there exist positive real numbers c_1, c_2, c_3 such that the quantities $a_{i1}c_1 + a_{i2}c_2 + a_{i3}c_3$ for $i = 1, 2, 3$ are either all positive, all negative, or all zero.

3. Let x, y, z be positive real numbers, not all equal, and define

$$a = x^2 - yz, \qquad b = y^2 - zx, \qquad c = z^2 - xy.$$

Express x, y, z in terms of a, b, c. (Hint: can you find a linear algebra interpretation of a, b, c, by making a certain matrix involving x, y, z?)

4. If A and B are square matrices of the same size such that $ABAB = 0$, does it follow that $BABA = 0$?

5. Suppose that A, B, C, D are $n \times n$ matrices (with entries in some field), such that AB^T and CD^T are symmetric, and $AD^T - BC^T = I$. Prove that $A^T D - C^T B = I$. (Hint: find a more "matricial" interpretation of the condition $AD^T - BC^T = I$.)

6. Let $x_1, \ldots, x_N$ be distinct unit vectors in $\mathbb{R}^n$ forming a regular simplex in projective space $\mathbb{RP}^{n-1}$, i.e. $|\langle x_i, x_j\rangle| = \alpha$ for some $0 \le \alpha < 1$. Show that $N \le n(n+1)/2$.

2 Determinants

7. Let D_n denote the value of the $(n-1)\times(n-1)$ determinant

$$\begin{vmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 6 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & n+1 \end{vmatrix}.$$

Is the set $\{D_n/n!\}_{n\ge 2}$ bounded?

8. Let p be a prime number. Prove that the determinant of the matrix

$$\begin{bmatrix} x & y & z \\ x^p & y^p & z^p \\ x^{p^2} & y^{p^2} & z^{p^2} \end{bmatrix}.$$

is congruent modulo p to a product of polynomials of the form $ax + by + cz$, where a, b, c are integers.

9. Let A be a $2n \times 2n$ skew-symmetric matrix (i.e., a matrix in which $A_{ij} = -A_{ji}$) with integer entries. Prove that the determinant of A is a perfect square. (Hint: prove a polynomial identity.)

10. Let x_i, $i = 1, \ldots, n$ and y_j, $j = 1, \ldots, n$ be $2n$ distinct real numbers. Calculate the determinant of the matrix whose (i, j) entry is $1/(x_i - y_j)$. Using this, show that the matrix whose entries are $1/(i+j-1)$ is invertible and that its inverse has integer entries.

11. Let A be the $n \times n$ matrix with $A_{jk} = \cos(2\pi(j + k)/n)$. Find the determinant of $I + A$.

12. Let A be a $2n \times 2n$ matrix, with entries chosen independently at random. Every entry is chosen to be 0 or 1, each with probability 1/2. Find the expected value of $\det(A - A^t)$ (as a function of n), where A^t is the transpose of A.

3 Eigenvalues and Related Things

13. Let $x_1, x_2, \ldots, x_n$ be differentiable (real-valued) functions of a single variable t which satisfy

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
\frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
&\vdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n
\end{aligned}$$

for some constants $a_{ij} > 0$. Suppose that for all i, $x_i(t) \to 0$ as $t \to \infty$. Are the functions $x_1, x_2, \ldots, x_n$ necessarily linearly dependent?

14. Let G be a finite set of real $n \times n$ matrices $\{M_i\}$, $1 \le i \le r$, which form a group under matrix multiplication. Suppose that $\sum_{i=1}^{r} \mathsf{tr}(M_i) = 0$, where $\mathsf{tr}(A)$ denotes the trace of the matrix A. Prove that $\sum_{i=1}^{r} M_i$ is the $n \times n$ zero matrix.

15. Let A be an $n \times n$ real symmetric matrix and B an $n \times n$ positive definite matrix. (A square matrix over $\mathbb{R}$ is *positive definite* if it is symmetric and all its eigenvalues are positive.) Show that all eigenvalues of AB are real. Hint. Use the following two facts from linear algebra: (a) all eigenvalues of a real symmetric matrix are real, and (b) a positive definite matrix has a positive definite square root.

16. Let A be an $n \times n$ matrix of real numbers for some $n \ge 1$. For each positive integer k, let $A^{[k]}$ be the matrix obtained by raising each entry to the kth power. Show that if $A^k = A^{[k]}$ for $k = 1, 2, \cdots, n+1$, then $A^k = A^{[k]}$ for all $k \ge 1$.

17. Let n be a positive integer. Suppose that A, B, and M are $n \times n$ matrices with real entries such that $AM = MB$, and such that A and B have the same characteristic polynomial. Prove that $\det(A - MX) = \det(B - XM)$ for every $n \times n$ matrix X with real entries.

4 Combinatorics

18. A mansion has n rooms. Each room has a lamp and a switch connected to its lamp. However, switches may also be connected to lamps in other rooms, subject to the following condition: if the switch in room a is connected to the lamp in room b, then the switch in room b is also connected to the lamp in room a. Each switch, when flipped, changes the state (from on to off or vice versa) of each lamp connected to it. Suppose at some points the lamps are all off. Prove that no matter how the switches are wired, it is possible to flip some of the switches to turn all of the lamps on. (Hint: interpret as a linear algebra problem over the field of two elements.)

19. Let n and k be positive integers. Say that a permutation σ of $\{1, 2, \ldots, n\}$ is *k-limited* if $|\sigma(i) - i| \le k$ for all i. Prove that the number of k-limited permutations of $\{1, 2, \ldots, n\}$ is odd if and only if $n \equiv 0$ or 1 (mod $2k + 1$).

20. Consider bipartite graphs where the vertices on each side are labeled $\{1, 2, \ldots, n\}$. Find the number of such graphs for which there are an odd number of perfect matchings.

21. Show that the edges of the complete graph on n vertices cannot be partitioned into fewer than $n-1$ edge-disjoint complete bipartite subgraphs.

5 Miscellaneous

22. Let p be a prime, and let $A = (a_{ij})_{i,j=0}^{p-1}$ be the $p \times p$ matrix defined by

$$a_{ij} = \binom{i+j}{i}, \quad 0 \le i, j \le p-1.$$

Show that $A^3 \equiv I \pmod{p}$, where I denotes the identity matrix. In other words, every entry of $A^3 - I$, evaluated over $\mathbb{Z}$, is divisible by p.

23. Let A be an $n \times n$ real matrix with every row and column sum equal to 0. Let $A[i,j]$ denote A with row i and column j removed. Show that $(-1)^{i+j} \det A[i,j]$ is independent of i and j. Can you express this determinant in terms of the eigenvalues of A?

24. Find the unique sequence $a_0, a_1, \ldots$ of real numbers such that for all $n \ge 0$ we have

$$\det \begin{bmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_n & a_{n+1} & \cdots & a_{2n} \end{bmatrix} = \det \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_2 & a_3 & \cdots & a_{n+1} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_n & a_{n+1} & \cdots & a_{2n-1} \end{bmatrix} = 1.$$

(When $n = 0$ the second matrix is empty and by convention has determinant one.)

25. Let $A = A(n)$ be the $n \times n$ real matrix given by

$$A_{ij} = \begin{cases} 1, & j = i+1 \ (1 \le i \le n-1) \\ 1, & j = i-1 \ (2 \le i \le n) \\ 0, & \text{otherwise.} \end{cases}$$

Let $V_n(x) = \det(xI - A)$, so $V_0(x) = 1$, $V_1(x) = x$, $V_2(x) = x^2 - 1$, $V_3(x) = x^3 - 2x$. Show that $V_{n+1}(x) = xV_n(x) - V_{n-1}(x)$, $n \ge 1$.

26. Show that

$$V_n(2\cos\theta) = \frac{\sin((n+1)\theta)}{\sin(\theta)}.$$

Deduce that the eigenvalues of $A(n)$ are $2\cos(j\pi/(n+1))$, $1 \le j \le n$.

27. Given $v = (v_1, \ldots, v_n)$ where each $v_i = 0$ or 1, let $f(v)$ be the number of even numbers among the n numbers

$$v_1 + v_2 + v_3, v_2 + v_3 + v_4, \ldots, v_{n-2} + v_{n-1} + v_n, v_{n-1} + v_n + v_1, v_n + v_1 + v_2.$$

For which positive integers n is the following true: for all $0 \le k \le n$, exactly $\binom{n}{k}$ vectors of the 2^n vectors $v \in \{0,1\}^n$ satisfy $f(v) = k$?

28. Let $M(n)$ denote the space of all real $n \times n$ matrices. Thus $M(n)$ is a real vector space of dimension n^2. Let $f(n)$ denote the maximum dimension of a subspace V of $M(n)$ such that every nonzero element of V is invertible.

    (a) (easy) Show that $f(n) \le n$.

(b) (fairly easy) Show that if n is odd, then $f(n) = 1$.

(c) (extremely difficult) For what n does $f(n) = n$?

(d) (even more difficult) Find a formula for $f(n)$ for all n.

29. (a) Let n be a positive integer. Prove that the matrix

$$\left(\frac{1}{(i+j)^2}\right)_{i,j=1}^{n}$$

is positive definite.

(b) Let $0 < a < b$. Prove that for any continuous function $f\colon [a,b] \to \mathbb{R}$,

$$\int_a^b \int_a^b \frac{f(x)f(y)}{(x+y)^2}\, dxdy \geq 0.$$

Remark. Feel free to use any of equivalent characterizations of what is meant by saying "a symmetric matrix $A = (a_{ij})_{i,j=1}^n$ is positive definite":

- All its eigenvalues are positive.
- For any $x \in \mathbb{R}^n, x \neq 0$, we have $x^T A x > 0$.
- (Sylvester's criterion) All its principal minors are positive. Principal minors are the values $\det(a_{ij})_{i,j\in I}$, for $I \subseteq \{1,\ldots,n\}$.

30. Define $A_0 = \begin{pmatrix} 0 \end{pmatrix}$, $A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

$$A_{n+1} = \begin{pmatrix} A_n & I_{2^n} \\ I_{2^n} & A_n \end{pmatrix},$$

where I_m is the $m \times m$ identity matrix. Prove that A_n has $n+1$ distinct eigenvalues with multiplicities $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$.

31. Let $f(z) = z^{n-1} + c_{n-2}z^{n-2} + \ldots + c_1 z + c_0$ be a polynomial with complex coefficients, such that $c_0 c_{n-2} = c_1 c_{n-3} = \ldots = 0$. Prove that $f(z)$ and $z^n - 1$ have at most $n - \sqrt{n}$ common roots.

32. Let Q be an n-by-n real orthogonal matrix, and let $u \in \mathbb{R}^n$ be a unit column vector (that is, $u^T u = 1$). Let $P = I - 2uu^T$, where I is the n-by-n identity matrix. Show that if 1 is not an eigenvalue of Q, then 1 is an eigenvalue of PQ.