PROBLEMS ON SUMS AND INTEGRALS

Contents

1 An example

1.1 Background

This example is taken from *Proofs from THE BOOK*, which contains many wonderful tricks of summation and integrals. It is definitely worth a read before the Putnam.

We have all heard about the Riemann Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Our task today is to explicitly find its values at the *even positive integers*. Evaluating this function at any other points is extremely difficult: with great effort people finally showed that $\zeta(3)$ is irrational, and little else is known. In 18.112 you can explore how to define this function everywhere, and how to evaluate this function at the negative integers.

1.2 An Identity

Our method of finding this sum comes from an unexpected place: the following identity about a trigonometry function

$$\pi \cot \pi x = \sum_{n \in \mathbb{Z}} \frac{1}{x+n}$$

We first note that the sum, as presented, doesn't quite make sense: it diverges. The correct way of presenting the sum is

$$\frac{1}{x} + \sum_{n \in \mathbb{N}} \frac{1}{x+n} + \frac{1}{x-n} = \frac{1}{x} + \sum_{n \in \mathbb{N}} \frac{2x}{x^2 - n^2}$$

Now the sum converges, and it satisfies two other important properties: it *converges absolutely*, and *converges uniformly* on any compact interval minus the integers. If you are unfamiliar with these two concepts, I strongly recommend reading *Principles of Mathematical Analysis* by Walter Rudin, which contains a nice exposition.

Now we are supposed to prove the identity. Euler did this by a fancy bash; however, a more beautiful proof was discovered by Herglotz, and properly named *Herglotz Trick*.

Proof. Let f, g be functions equal to LHS and RHS of the equation. We want prove that they are equal; we achieve this by proving several properties of the function $h = f - g$.

(1) The function $h = f - g$ is defined everywhere, and continuous everywhere. We only need to show this on a neighbor of 0, and the rest is analogous. We first note that $\pi \cot \pi x - \frac{1}{x}$ is continuous in the neighbor(this can be shown by Taylor series, or straight from definition). Then we note that the rest of the sum converges uniformly, and thus is also continuous.

(2) The function $h = f - g$ is periodic of period 1. This can be easily seen from the expression.

(3) The function $h = f - g$ satisfy the functional equation

$$2h(x) = h(\frac{x}{2}) + h(\frac{x+1}{2}) \tag{1}$$

We can verify this for f and g individually at the non-integer points(note there is an interchange of summation on LHS, which needs to be justified by absolute convergence). The equations carry to the integer points by continuity.

Now we use the trick: we note that any periodic, continuous function that satisfy (1) must be a constant! In fact, $|h|$ must assume maximum value at some point x_0. However, from (1) we find that if $|h|$ has maximum at x_0, then it must assume maximum at $\frac{x_0}{2}$ and $\frac{x_0+1}{2}$ as well. Repeating the argument, we find that $|h|$ assumes maximum at a series of points approaching 0, thus it is equal to $|h(0)|$ everywhere. We conclude that $h = 0$, by noting that $|h(0)| = 0$. □

1.3 Application to the Riemann Zeta Function

Now some of you might be able to see the connection between the identity and the zeta function.

$$\pi \cot \pi x - \frac{1}{x} = \sum_{n \in \mathbb{N}} \frac{2x}{x^2 - n^2}$$

We consider x in a neighbor of 0. Expanding RHS,

$$\sum_{n \in \mathbb{N}} \frac{2x}{x^2 - n^2} = \sum_{n \in \mathbb{N}} \sum_{j=1}^{\infty} -\frac{2}{n^{2j}} x^{2j-1}$$

We can swap the sum by absolute convergence

$$\sum_{n \in \mathbb{N}} \sum_{j=1}^{\infty} -\frac{2}{n^{2j}} x^{2j-1} = -\sum_{j=1}^{\infty} 2\zeta(2j) x^{2j-1}$$

Note this is precisely the Taylor Expansion! With one formula, we have captured the values of the ζ function at all the positive even points!

All is left is to find the Taylor expansion of $\pi \cot \pi x - \frac{1}{x}$ at a neighbor of 0. But this is very easy: we can do it by direct differentiation, or we can do it smartly by noting that

$$\pi \cot \pi x - \frac{1}{x} = \frac{\pi x \cos \pi x - \sin \pi x}{x \sin \pi x}$$

Thus, if we let

$$\pi \cot \pi x - \frac{1}{x} = \sum_{j=1}^{\infty} a_j x^{2j-1}$$

Then the series a_j satisfy the relation

$$\sum_{i=0}^{j-1} \frac{(-1)^i \pi^{2i+1}}{(2i+1)!} a_{j-i} = (-1)^j \pi^{2j+1} (\frac{1}{(2j)!} - \frac{1}{(2j+1)!})$$

A convention is to write

$$\zeta(2j) = \frac{B_j \pi^{2j}}{(2j)!}$$

Thus the recursion becomes

$$-2 \sum_{i=0}^{j-1} \frac{(-1)^i}{(2i+1)!(2j-2i)!} B_{j-i} = (-1)^j (\frac{1}{(2j)!} - \frac{1}{(2j+1)!})$$

cleaning up,

$$\sum_{i=1}^{j} (-1)^i \binom{2j+1}{2i} B_i = -j$$

We can calculate some values, just for fun

$$\zeta(2) = \frac{\pi^2}{6}, \zeta(4) = \frac{\pi^4}{90}, \zeta(6) = \frac{\pi^6}{945} \cdots$$

2 Problems

2.1 Concept Problems

1. Compute
$$\lim_{n\to\infty} \left[\frac{1}{\sqrt{4n^2-1^2}} + \frac{1}{\sqrt{4n^2-2^2}} + \cdots + \frac{1}{\sqrt{4n^2-n^2}} \right].$$

2. Find
$$\lim_{n\to\infty} \frac{1}{n} \sum_{a=1}^{n} \sum_{b=1}^{n} \frac{a}{a^2+b^2}.$$

3. Show that $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$.

4. Find
$$\int_0^1 \log x \log(1-x) dx$$

5. Exhibit a sequence a_{ij} indexed by $\mathbb{Z}^2$ such that
$$\sum_i (\sum_j a_{ij}) \neq \sum_j (\sum_i a_{ij})$$
with all sums converging.

6. Exhibit a smooth(i.e. infinitely differentiable) function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 1$ when $x < 0$ yet $f(x) = 0$ when $x > 1$.(This function is one of the most important gadgets in Analysis)

2.2 Putnam-Style Problems

1. Evaluate the improper integral
$$\int_0^1 \frac{\log(1-x)}{x}\,dx.$$

2. Determine the value of the improper integral
$$\int_0^\infty \frac{x}{e^x-1}\,dx.$$

3. (a) Show that that $\min(a,b) = \int_0^\infty \mathbf{1}_{\le a}(t)\mathbf{1}_{\le b}(t)\,dt$ for any nonnegative real numbers $a, b \ge 0$. (What do you think $\mathbf{1}_{\le c}(t)$ means?)

   (b) Show that if $r_1, \ldots, r_n$ are nonnegative reals and $x_1, \ldots, x_n$ are real numbers then
$$\sum_{i=1}^n \sum_{j=1}^n \min(r_i, r_j) x_i x_j \ge 0.$$

4. Evaluate the following:
$$\int_0^\infty \left(x - \frac{x^3}{2} + \frac{x^5}{2\cdot 4} - \frac{x^7}{2\cdot 4\cdot 6} + \cdots\right)\left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} + \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \cdots\right)dx.$$

5. Show that
$$\int_0^1 x^{-x}\,dx = \sum_{n\ge 1} n^{-n}.$$

6. Suppose that f is a function on the interval $[1,3]$ such that $-1 \le f(x) \le 1$ for all x and $\int_1^3 f(x)\,dx = 0$. Determine the largest possible value of
$$\int_1^3 \frac{f(x)}{x}\,dx.$$

7. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and satisfy $f(x) \ge 1$ for all x. Suppose that
$$f(x)f(2x)\ldots f(nx) \le 2018n^{2019}$$
for every positive integer n and $x \in \mathbb{R}$. Must f be constant?

8. A rectangle in $\mathbb{R}^2$ is called *great* if either its width or height is an integer. Prove that if a rectangle X can be dissected into great rectangles, then the rectangle X is itself great.

9. Compute
$$\sum_{k\ge 0} \frac{2^k}{5^{2^k}+1}.$$

10. Prove that

$$\lim_{n\to\infty}\left(\prod_{k=0}^{n}\binom{n}{k}\right)^{\frac{1}{n(n+1)}}=\sqrt{e}.$$

11. Let $f\colon \mathbb{R}_{\geq 0}\to\mathbb{R}$ be a strictly decreasing continuous function such that $\lim_{x\to\infty} f(x)=0$. Prove that

$$\int_0^\infty \frac{f(x)-f(x+1)}{f(x)}\,dx$$

diverges.

12. A rectangular prism X is contained within a rectangular prism Y.

   (a) Is it possible the surface area of X exceeds that of Y?

   (b) Is it possible the sum of the 12 side lengths of X exceeds that of Y?

13. For all $n\geq 1$, let

$$a_n=\sum_{k=1}^{n-1}\frac{\sin\frac{2k-1}{2n}\pi}{\cos^2\frac{k-1}{2n}\pi\cos^2\frac{k}{2n}\pi}.$$

Find

$$\lim_{n\to\infty}\frac{a_n}{n^3}.$$

14. For $a,b,c>0$ prove that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{4}{a+b}+\frac{4}{b+c}+\frac{4}{c+a}\geq\frac{12}{3a+b}+\frac{12}{3b+c}+\frac{12}{3c+a}.$$

15. Evaluate

$$\lim_{x\to 1^-}\prod_{n\geq 0}\left(\frac{1+x^{n+1}}{1+x^n}\right)^{x^n}.$$

16. Suppose that $f\colon[0,1]^2\to\mathbb{R}$ is continuous. Show that

$$\int_0^1\left(\int_0^1 f(x,y)dx\right)^2dy+\int_0^1\left(\int_0^1 f(x,y)dy\right)^2dx$$
$$\leq\left(\int_0^1\int_0^1 f(x,y)\,dx\,dy\right)^2+\int_0^1\int_0^1[f(x,y)]^2\,dx\,dy.$$

17. For each positive integer k, let $A(k)$ be the number of odd divisors of k in the interval $\left[1,\sqrt{2k}\right)$. Evaluate:

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{A(k)}{k}.$$

18. (a) Let $f(x,y,z)$ be a continuous real-valued function on $\mathbb{R}^3$. Suppose that for every sphere S of radius 1, the integral of $f(x,y,z)$ over the surface of S equals 0. Must $f(x,y,z)$ be identically 0?

    (b) What if f is required to be smooth and of compact support?

2.3 Problems related to more advanced topics

1. For positive integer n, let $p(n)$ be the number of ways to partition n into the sum of some positive integers, quotient permutation of the integers. The sequence $p(n)$ starts $1, 2, 3, 5, 7, 11, \ldots$.

   (a) Prove that
   $$\lim_{n\to\infty} \frac{\log p(n)}{\sqrt{n}}$$
   exists, and find its value.

   (b)(Hard) Let the answer of (a) be C. Prove that
   $$\lim_{n\to\infty} \frac{np(n)}{e^{C\sqrt{n}}}$$
   exists.

   (c)(Hard)Determine the value of the limit in (b).

2. Prove that if f is a continuous function on the unit circle S^1 parametrized by $\theta \in [0, 2\pi)$, θ_0 is an angle such that $\frac{\theta_0}{2\pi}$ is irrational, then
   $$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} f(m\theta_0) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta.$$
   Hint: Fourier expansion does not hold here, but can you do something similar?

3. Let $f(x)$ be a smooth function with *compact support* on $\mathbb{R}$, and $\hat{f}$ be its Fourier transform
   $$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi x \xi i} dx.$$
   Prove the Poisson Summation Formula
   $$\sum_{n\in\mathbb{Z}} \hat{f}(n) = \sum_{n\in\mathbb{Z}} f(n).$$
   Using this result, prove that for all $t > 0$
   $$\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} = \frac{1}{\sqrt{t}} \sum_{n\in\mathbb{Z}} e^{-\frac{\pi n^2}{t}}.$$
   This formula is useful in the analytic extension of the Riemann Zeta function.

4. Prove the Jacobi Triple Product Formula for $|q| < 1$
   $$\prod_{m=1}^{\infty} (1 - q^{2m})(1 + \omega^2 q^{2m-1})(1 + \omega^{-2} q^{2m-1}) = \sum_{n=-\infty}^{\infty} \omega^{2n} q^{n^2}.$$
   Using this, either find the number of integer solutions for $12345654321 = a^2 + b^2$, or exhibit the difference between the number of even and odd partitions of n(a partition $n = a_1 + \cdots + a_k$ is even iff k is even).

5. A smooth function $f : \mathbb{C} \to \mathbb{C}$ is *harmonic* if

$$\Delta f := (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})f = 0.$$

(1) Prove the Mean Value Property of Harmonic functions: for any $z \in \mathbb{C}$ and $r > 0$, we have

$$\int_0^{2\pi} f(z + e^{i\theta} r) d\theta = 2\pi f(z).$$

(2) Find all harmonic functions on $\mathbb{C}$ that is real and positive everywhere.

(3)(Hard) Harmonic functions can be similarly defined on any open subset of the plane. Find all harmonic functions on $\mathbb{C}\backslash\{0\}$ that is real and positive on its domain.

6. (Hard) Let

$$f(z) = \sum_{i=0}^{d} c_i z^i$$

be a polynomial with strictly positive coefficients($c_i > 0$ for all $0 \leq i \leq d$). Let $A_{n,k}$ be the coefficient of z^n in $f^k(z)$. Prove that there exists some $K > 0$(possibly depending on f) such that for all $k > K$ and $1 \leq n \leq dk - 1$, we have

$$A_{n,k}^2 \geq A_{n-1,k} A_{n+1,k}.$$