PROBLEMS ON RECURRENCES

1. Let $T_0 = 2, T_1 = 3, T_2 = 6$, and for $n \geq 3$,
$$T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3}.$$
The first few terms are: $2, 3, 6, 14, 40, 152, 784, 5168, 40576, 363392$. Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$, where $\{A_n\}$ and $\{B_n\}$ are well-known sequences.

2. Define u_n by $u_0 = 0, u_1 = 4$, and $u_{n+2} = \frac{6}{5}u_{n+1} - u_n$. Show that $|u_n| \leq 5$ for all n.

3. Prove or disprove that there exists a positive real number u such that $\lfloor u^n \rfloor - n$ is an even integer for all positive integers n.

4. Show that the next integer above $(\sqrt{3}+1)^{2n}$ is divisible by 2^{n+1}.

5. Let n be a positive integer and let $a_1, \ldots, a_{n-1}$ be arbitrary real numbers. Define the sequences $u_0, \ldots, u_n$ and $v_0, \ldots, v_n$ inductively by $u_0 = u_1 = v_0 = v_1 = 1$, and $u_{k+1} = u_k + a_k u_{k-1}$, $v_{k+1} = v_k + a_{n-k} v_{k-1}$ for $k = 1, \ldots, n-1$.
Prove that $u_n = v_n$.

6. Let $a_0, a_1, \ldots$ be an arbitrary sequence of positive integers, and $p_0 = 1, q_0 = 0, p_1 = a_0, q_1 = 1$. Consider the recurrence
$$p_{n+2} = a_{n+1}p_{n+1} + p_n,$$
$$q_{n+2} = a_{n+1}q_{n+1} + q_n.$$
Show that p_n, q_n are coprime for any $n \geq 0$.

7. Let $Q_0(x) = 1$, $Q_1(x) = x$, and
$$Q_n(x) = \frac{(Q_{n-1}(x))^2 - 1}{Q_{n-2}(x)}$$
for all $n \geq 2$. Show that, whenever n is a positive integer, $Q_n(x)$ is equal to a polynomial with integer coefficients.

8. For a positive integer n and any real number c , define x_k recursively by $x_0 = 0, x_1 = 1$, and for $k \geq 0$,
$$x_{k+2} = \frac{cx_{k+1} - (n-k)x_k}{k+1}.$$
Fix n and then take c to be the largest value for which $x_{n+1} = 0$. Find x_k in terms of n and $k, 1 \leq k \leq n$.

9. Solve the recurrence
$$(n+1)(n+2)a_{n+2} - 3(n+1)a_{n+1} + 2a_n = 0,$$
with the initial conditions $a_0 = 2, a_1 = 3$.

10. Define $u_0 = 1$ and for $n \geq 0$,

$$2u_{n+1} = \sum_{k=0}^{n} \binom{n}{k} u_k u_{n-k}.$$

Find a simple expression for u_n.

11. Given some distinct positive integers $a_1, \ldots, a_n$. Two players are playing a game where there are m stones on the table at the beginning, and each player takes turn to choose a number i from 1 to n and take a_i stones from the table. A player loses if there is no valid move, i.e. there are not $\min a_i$ stones to take away from the table. Let $f(m)$ be 1 if the first player has a winning strategy, and 0 if the second player has a winning strategy. Show that f is eventually periodic.

12. Let $a_1 < a_2$ be two given integers. For any integer $n \geq 3$, let a_n be the smallest integer which is larger than a_{n-1} and can be uniquely represented as $a_i + a_j$, where $1 \leq i < j \leq n-1$. Given that there are only a finite number of even numbers in $\{a_n\}$, prove that the sequence $\{a_{n+1} - a_n\}$ is eventually periodic, i.e. that there exist positive integers T, N such that for all integers $n > N$, we have

$$a_{T+n+1} - a_{T+n} = a_{n+1} - a_n.$$

13. Let $1, 2, 3, \ldots, 2005, 2006, 2007, 2009, 2012, 2016, \ldots$ be a sequence defined by $x_k = k$ for $k = 1, 2, \ldots, 2006$ and $x_{k+1} = x_k + x_{k-2005}$ for $k \geq 2006$. Show that the sequence has 2005 consecutive terms each divisible by 2006.

14. Let $a_0, a_1, \ldots$ be a sequence such that $a_0 = 1, a_1 = 2$ and $a_{n+2} = 4a_{n+1} - a_n$ for all $n \geq 0$. Show that $a_m | a_{(2k+1)m}$ for all nonnegative integers m, k.

15. Let $a_0 = 2, a_1 = -3$ and for all $n \geq 2$,

$$a_n = \sum_{i=0}^{n-1} (2i-1) a_{n-i-1}.$$

Find a closed form for a_n in terms of n.

16. Solve the recurrence: $a_0 = 1, a_1 = 2019$ and

$$a_{n+2} a_n = a_{n+1}(a_n + a_{n+1})$$

for all $n \geq 0$.

17. Solve the first order recursion given by $x_0 = 1$ and $x_n = 1 + (1/x_{n-1})$. Does $\{x_n\}$ approach a limiting value as n increases?

18. Let $a_0 = 5/2$ and $a_k = a_{k-1}^2 - 2$ for $k \geq 1$. Compute

$$\prod_{i=0}^{\infty} \left(1 - \frac{1}{a_i}\right)$$

in closed form.

19. Given a real number a, we define a sequence by $x_0 = 1$, $x_1 = x_2 = a$, and $x_{n+1} = 2x_n x_{n-1} - x_{n-2}$ for $n \geq 2$. Prove that if $x_n = 0$ for some n, then the sequence is periodic.

20. Let k be an integer greater than 1. Suppose that $a_0 > 0$, and define

$$a_{n+1} = a_n + \frac{1}{\sqrt[k]{a_n}}$$

for $n > 0$. Evaluate

$$\lim_{n \to \infty} \frac{a_n^{k+1}}{n^k}.$$

21. Let $a_0, a_1, \ldots$ be any sequence of integers where $a_0 = 0$ and

$$a_{n+2} = c a_{n+1} + d a_n$$

for all $n \geq 0$ where c, d are some integers. Show that for any prime p that is not a factor of d, there exists $1 \leq i \leq p + 1$ such that $p | a_i$.

If $c = 6$ and $d = -1$, show furthermore that for any odd prime p there exists $1 \leq i < p$ such that $p | a_i$.

22. Let $a_0, a_1, \ldots$ be a sequence of integers where $a_0 = 0$ and

$$a_{n+2} = c a_{n+1} + a_n$$

for all $n \geq 0$ where c is a given integer. By pigeon hole principle, one can show that for any prime p, there exists $T \leq p^2$ such that

$$a_{n+T} \equiv a_n \mod p$$

holds for all $n \geq 0$. Show a much stronger result: for any prime p that is sufficiently large, there exists $T \leq 2p + 2$ such that

$$a_{n+T} \equiv a_n \mod p \quad \forall n \geq 0.$$

23. (*very* difficult) Let $a_0, a_1, \ldots$ satisfy a homogeneous linear recurrence (of finite degree) with constant coefficients. I.e., for some complex (or real, if you prefer) numbers $\nu_1, \ldots, \nu_k$ we have

$$a_n = \nu_1 a_{n-1} + \cdots + \nu_k a_{n-k}$$

for all $n \geq k$. Define

$$b_n = \begin{cases} 1, & a_n \neq 0 \\ 0, & a_n = 0. \end{cases}$$

Show that b_n is eventually periodic, i.e., there exists $p > 0$ such that $b_n = b_{n+p}$ for all n sufficiently large.