PROBLEMS ON POLYNOMIALS

Note. The terms "root" and "zero" of a polynomial are synonyms.

1. Find the cubic equation whose roots are the cubes of the roots of

$$x^3 + ax^2 + bx + c = 0.$$

2. (a) Determine all rational values for which a, b, c are the roots of

$$x^3 + ax^2 + bx + c = 0.$$

   (b) Show that the only real polynomials $\prod_{i=0}^{n-1}(x - a_i) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ in addition to those given by (a) are x^n, $x^2 + x - 2$, and exactly two others, which are approximately equal to

$$x^3 + .56519772x^2 - 1.76929234x + .63889690$$

   and

$$x^4 + x^3 - 1.7548782x^2 - .5698401x + .3247183.$$

3. Assuming that all the roots of the cubic equation $x^3 + ax^2 + bx + c$ are real, show that the difference between the greatest and the least roots is not less than $\sqrt{a^2 - 3b}$ nor greater than $2\sqrt{(a^2 - 3b)/3}$.

4. The nonconstant polynomials $P(z)$ and $Q(z)$ with complex coefficients have the same set of numbers for their zeros but possibly different multiplicities. The same is true of the polynomials $P(z) + 1$ and $Q(z) + 1$. Prove that $P(z) = Q(z)$. (On the original Exam, the assumption that $P(z)$ and $Q(z)$ are nonconstant was inadvertently omitted.)

5. If $a_0, a_1, \ldots, a_n$ are real numbers satisfying

$$\frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0,$$

   show that the equation $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0$ has at least one real root.

6. Determine all polynomials of the form

$$\sum_0^n a_i x^{n-i} \quad \text{with } a_i = \pm 1$$

   $(0 \le i \le n,\ \ 1 \le n < \infty)$ such that each has only real zeros.

7. Let $P(x)$ be a polynomial with real coefficients and form the polynomial

$$Q(x) = (x^2 + 1)P(x)P'(x) + x(P(x)^2 + P'(x)^2).$$

   Given that the equation $P(x) = 0$ has n distinct real roots exceeding 1, prove or disprove that the equation $Q(x) = 0$ has at least $2n - 1$ distinct real roots.

8. Prove that if

$$11z^{10} + 10iz^9 + 10iz - 11 = 0,$$

   then $|z| = 1$. (Here z is a complex number and $i^2 = -1$.)

9. Is there an infinite sequence $a_0, a_1, a_2, \ldots$ of nonzero real numbers such that for each $n = 1, 2, 3, \ldots$ the polynomial

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

has exactly n distinct real roots?

10. Find all real polynomials $p(x)$ of degree $n \geq 2$ for which there exist real numbers $r_1 < r_2 < \cdots < r_n$ such that

(i) $p(r_i) = 0, \quad i = 1, 2, \ldots, n,$

and

(ii) $p'\left(\frac{r_i + r_{i+1}}{2}\right) = 0, \quad i = 1, 2, \ldots, n-1,$

where $p'(x)$ denotes the derivative of $p(x)$.

11. (a) Let k be the smallest positive integer with the following property:

> There are distinct integers m_1, m_2, m_3, m_4, m_5 such that the polynomial $p(x) = (x - m_1)(x - m_2)(x - m_3)(x - m_4)(x - m_5)$ has exactly k nonzero coefficients.

Find, with proof, a set of integers m_1, m_2, m_3, m_4, m_5 for which this minimum k is achieved.

(b) Let $P(x) = x^{11} + a_{10}x^{10} + \cdots + a_0$ be a monic polynomial of degree eleven with real coefficients a_i, with $a_0 \neq 0$. Suppose that all the zeros of $P(x)$ are real, i.e., if α is a complex number such that $P(\alpha) = 0$, then α is real. Find (with proof) the least possible number of nonzero coefficients of $P(x)$ (including the coefficient 1 of x^{11}).

12. Let $P(x)$ be a polynomial of degree n such that $P(x) = Q(x)P''(x)$, where $Q(x)$ is a quadratic polynomial and $P''(x)$ is the second derivative of $P(x)$. Show that if $P(x)$ has at least two distinct roots then it must have n distinct roots.

13. (a) Let $p(z)$ be a polynomial of degree n, all of whose zeros have absolute value 1 in the complex plane. Put $g(z) = p(z)/z^{n/2}$. Show that all zeros of $g'(z) = 0$ have absolute value 1.

(b) Let $f(t) = \sum_{j=1}^{N} a_j \sin(2\pi j t)$, where each a_j is real and a_N is not equal to 0. Let N_k denote the number of zeros (including multiplicities) of $\frac{d^k f}{dt^k}$ in the half-open interval $[0, 1)$. Prove that

$$N_0 \leq N_1 \leq N_2 \leq \cdots \quad \text{and} \quad \lim_{k \to \infty} N_k = 2N.$$

14. For every non-constant polynomial p, let $H_p = \{z \in \mathbb{C} : |p(z)| = 1\}$. Prove that if $H_p = H_q$ for some polynomials p, q, then there exists a polynomial r such that $p = r^m$ and $q = \xi r^n$ for some positive integers m, n and constant $|\xi| = 1$.

15. For each integer m, consider the polynomial

$$P_m(x) = x^4 - (2m + 4)x^2 + (m - 2)^2.$$

For what values of m is $P_m(x)$ the product of two nonconstant polynomials with integer coefficients?

16. Let k be a fixed positive integer. The n-th derivative of $1/(x^k - 1)$ has the form $P_n(x)/(x^k - 1)^{n+1}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$.

17. Let p be a prime number. Prove that the determinant of the matrix

$$\begin{pmatrix} x & y & z \\ x^p & y^p & z^p \\ x^{p^2} & y^{p^2} & z^{p^2} \end{pmatrix}$$

is congruent modulo p to a product of polynomials of the form $ax + by + cz$, where a, b, c are integers. (We say two integer polynomials are congruent modulo p if corresponding coefficients are congruent modulo p.)

18. Let $f(z) = az^4 + bz^3 + cz^2 + dz + e = a(z - r_1)(z - r_2)(z - r_3)(z - r_4)$ where a, b, c, d, e are integers, $a \neq 0$. Show that if $r_1 + r_2$ is a rational number and $r_1 + r_2 \neq r_3 + r_4$, then $r_1 r_2$ is a rational number.

19. Let $P(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_0$ be a polynomial with integer coefficients. Suppose that r is a rational number such that $P(r) = 0$. Show that the n numbers

$$c_n r, c_n r^2 + c_{n-1} r, c_n r^3 + c_{n-1} r^2 + c_{n-2} r, \ldots, c_n r^n + c_{n-1} r^{n-1} + \cdots + c_1 r$$

are integers.

20. Let n be a positive integer. Find the number of pairs P, Q of polynomials with real coefficients such that

$$(P(X))^2 + (Q(X))^2 = X^{2n} + 1$$

and $\deg P > \deg Q$.

21. Let k be a positive integer. Prove that there exist polynomials $P_0(n), P_1(n), \ldots, P_{k-1}(n)$ (which may depend on k) such that for any integer n,

$$\left\lfloor \frac{n}{k} \right\rfloor^k = P_0(n) + P_1(n) \left\lfloor \frac{n}{k} \right\rfloor + \cdots + P_{k-1}(n) \left\lfloor \frac{n}{k} \right\rfloor^{k-1}.$$

($\lfloor a \rfloor$ means the largest integer $\leq a$.)

22. Find the smallest positive integer j such that for every polynomial $p(x)$ with integer coefficients and for every integer k, the integer

$$p^{(j)}(k) = \left. \frac{d^j}{dx^j} p(x) \right|_{x=k}$$

(the j-th derivative of $p(x)$ at k) is divisible by 2016.

23. Let n be a positive integer. Show that there are positive real numbers $a_0, a_1, \ldots, a_n$ such that for each choice of signs the polynomial

$$\pm a_n x^n \pm a_{n-1} x^{n-1} \pm \cdots \pm a_1 x \pm a_0$$

has n distinct real roots.

24. Determine all pairs $P(x), Q(x)$ of complex polynomials with leading coefficient 1 such that $P(x)$ divides $Q(x)^2 + 1$ and $Q(x)$ divides $P(x)^2 + 1$.

25. Let $p(x)$ be a nonconstant polynomial with real coefficients. For every positive integer n, let

$$q_n(x) = (x+1)^n p(x) + x^n p(x+1).$$

Prove that there are only finitely many numbers n such that all roots of $q_n(x)$ are real.

26. Let $ax^3 + bx^2 + cx + d$ be a polynomial with three distinct real roots. How many real roots are there of the equation

$$4(ax^3 + bx^2 + cx + d)(3ax + b) = (3ax^2 + 2bx + c)^2?$$

27. Does there exist a finite set M of nonzero real numbers, such that for any positive integer n, there exists a polynomial of degree at least n with all coefficients in M, all of whose roots are real and belong to M?

28. Suppose that the polynomial $ax^2 + (c-b)x + (e-d)$ has two real roots, both greater than 1. Prove that $ax^4 + bx^3 + cx^2 + dx + e$ has at least one real root.

29. Suppose that $a, b, c \in \mathbb{C}$ are such that the roots of the polynomial $z^3 + az^2 + bz + c$ all satisfy $|z| = 1$. Prove that the roots of $x^3 + |a|x^2 + |b|x + |c|$ all satisfy $|x| = 1$.

30. Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a monic polynomial of degree n with complex coefficients a_i. Suppose that the roots of $P(x)$ are $x_1, x_2, \cdots, x_n$, i.e., we have $P(x) = (x - x_1)(x - x_2)\cdots(x - x_n)$. The *discriminant* $\Delta(P(x))$ is defined by

$$\Delta(P(x)) = \prod_{1 \le i < j \le n} (x_i - x_j)^2.$$

Show that

$$\Delta(x^n + ax + b) = (-1)^{\binom{n}{2}} \left(n^n b^{n-1} + (-1)^{n-1}(n-1)^{n-1}a^n\right).$$

HINT. First note that

$$P^{'}(x) = P(x)\left(\frac{1}{x - x_1} + \cdots + \frac{1}{x - x_n}\right).$$

Use this formula to establish a connection between $\Delta(P(x))$ and the values $P^{'}(x_i)$, $1 \le i \le n$.

31. Let $P_n(x) = (x+n)(x+n-1)\cdots(x+1) - (x-1)(x-2)\cdots(x-n)$. Show that all the zeros of $P_n(x)$ are purely imaginary, i.e., have real part 0.

32. Let $P(x)$ be a polynomial with complex coefficients such that every root has real part a. Let $z \in \mathbb{C}$ with $|z| = 1$. Show that every root of the polynomial $R(x) = P(x-1) - zP(x)$ has real part $a + \frac{1}{2}$.

33. Let $d \ge 1$. It is not hard to see that there exists a polynomial $A_d(x)$ of degree d such that

$$F_d(x) := \sum_{n \ge 0} n^d x^n = \frac{A_d(x)}{(1-x)^{d+1}}. \tag{1}$$

For instance, $A_1(x) = x$, $A_2(x) = x + x^2$, $A_3(x) = x + 4x^2 + x^3$. Show that every root of $A_d(x)$ is real. HINT. First differentiate equation (1).

34. Let $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ be a monic polynomial with complex coefficients. Choose $j \in \{0, \ldots, n\}$ so that the roots of P can be labeled $\alpha_1, \ldots, \alpha_n$ with

$$|\alpha_1|, \ldots, |\alpha_j| > 1, \qquad |\alpha_{j+1}|, \ldots, |\alpha_n| \leq 1.$$

Prove that

$$\prod_{i=1}^{j} |\alpha_i| \leq \sqrt{|a_0|^2 + \cdots + |a_{n-1}|^2 + 1}.$$

Hint. One approach is to deduce this from an identity involving the polynomials $(z - \alpha_1) \cdots (z - \alpha_j)$ and $(\alpha_{j+1}z - 1) \cdots (\alpha_n z - 1)$.

35. Let $Q(x)$ be any monic polynomial of degree n with real coefficients. Prove that

$$\sup_{x \in [-2,2]} |Q(x)| \geq 2.$$

Hint. Let $P_n(x)$ be the monic polynomial satisfying

$$P_n(2\cos\theta) = 2\cos(n\theta) \qquad (\theta \in \mathbb{R}),$$

and examine the values of $P_n(x) - Q(x)$ at points where $|P_n(x)| = 2$.

Optional. Prove that equality only holds for $Q = P_n$.

36. Let $P(x), Q(x)$ be two polynomials with all real roots $r_1 \leq r_2 \leq \cdots \leq r_n$ and $s_1 \leq s_2 \leq \cdots \leq s_{n-1}$, respectively. We say that $P(x)$ and $Q(x)$ are *interlaced* if

$$r_1 \leq s_1 \leq r_2 \leq s_2 \leq \cdots \leq s_{n-1} \leq r_n.$$

Prove that $P(x)$ and $Q(x)$ are interlaced if and only if the polynomial $P + tQ$ has all real roots for all $t \in \mathbb{R}$.

37. Let $P(x)$ be a polynomial with real coefficients. For $t \in \mathbb{R}$, let $V(P, t)$ denote the number of sign changes in the sequence

$$P(t), P'(t), P''(t), \ldots.$$

(A *sign change* in a sequence is a pair of terms, one positive and one negative, with only zeros in between.) Prove that for any $a, b \in \mathbb{R}$, the number of roots of P in the half-open interval $(a, b]$, counted with multiplicities, is equal to $V(P, a) - V(P, b)$ minus a nonnegative even integer. Then deduce Descartes's rule of signs as a corollary.

38. Let $P(x)$ be a squarefree polynomial with real coefficients. Define the sequence of polynomials $P_0, P_1, \ldots$ by setting $P_0 = P$, $P_1 = P'$, and

$$P_{i+2} = -\mathrm{rem}(P_i, P_{i+1}),$$

where $\mathrm{rem}(A, B)$ means the remainder upon Euclidean division of A by B; upon arriving at a nonzero constant polynomial P_r, stop. Prove that for any $a, b \in \mathbb{R}$, the number of zeros of P in $(a, b]$ is $\sigma(a) - \sigma(b)$, where $\sigma(t)$ is the number of sign changes in the sequence

$$P_0(t), P_1(t), \ldots, P_r(t).$$