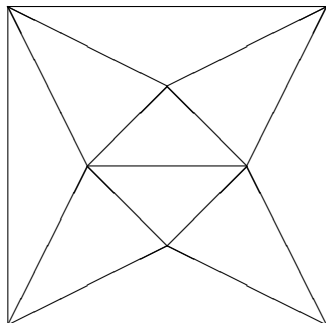


## 18.A34 PROBLEMS #11

123. [1] Can the following figure be drawn without lifting one's pencil from the paper or retracing any lines?



124. A *covering congruence* is a finite set  $(a_1, m_1), \dots, (a_k, m_k)$  of ordered pairs of integers, where  $0 < m_1 < \dots < m_k$ , such that every integer can be written in at least one of the forms  $m_i x + a_i$ , where  $x$  is an integer.
- (a) [1] Find a covering congruence with  $m_1 > 1$ .
- (b) [4] Does there exist a covering congruence with  $m_1$  arbitrarily large? (There exists one with  $m_1 = 40$  and  $k > 10^{50}$ .)
- (c) [5] Does there exist a covering congruence with each  $m_i$  odd?
- (d) [1.5] Show that in a covering congruence,

$$\frac{1}{m_1} + \dots + \frac{1}{m_k} \geq 1.$$

- (e) [2.5] Show that in a covering congruence,

$$\frac{1}{m_1} + \dots + \frac{1}{m_k} > 1.$$

This is equivalent [why?] to saying that there does not exist a covering congruence where every integer can be written *uniquely* in the form  $m_i x + a_i$ . In other words, the set  $\mathbb{Z}$  of integers cannot be written as a *disjoint* union of arithmetic progressions, all with different differences.

125. (a) [2] Evaluate the infinite product

$$\prod_{n=0}^{\infty} (1 + x^{2^n}), \quad |x| < 1.$$

(b) [3.5] Consider the Taylor series expansion

$$\prod_{n=0}^{\infty} (1 + x^{2^n} + x^{2^{n+1}}) = a_0 + a_1x + \cdots .$$

The sequence  $(a_0, a_1, \dots)$  begins

$$(1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, 5, 4, 7, 3, 8, \dots).$$

Show that every positive rational number occurs exactly once among the fractions  $a_{i+1}/a_i$ , and moreover these fractions are in lowest terms.

(c) [3] Let  $f(x) = 1/(1 + 2[x] - x)$ , and define a sequence  $x_0, x_1, \dots$  by  $x_0 = 1$  and  $x_{n+1} = f(x_n)$  for  $n \geq 0$ . Show that  $x_n = a_{n+1}/a_n$ .

126. [1]–[5] Color the  $(x, y)$ -plane in two colors, i.e., color every point in the plane either red or blue, say. Will there always exist two points at distance one which are the same color? What about three colors? Four colors? Etc.
127. [2.5] Let  $f(n)$  be a nonconstant polynomial with integer coefficients. Show that  $f(n)$  is composite for infinitely many values of  $n$ . (NOTE: It is not known whether there exists a polynomial  $g(n)$  of degree greater than one with integer coefficients such that  $g(n)$  is prime for infinitely many values of  $n$ . However, many polynomials, such as  $n^2 + 1$ , are conjectured to have this property.)
128. [2.5] What positive real numbers  $a$  have the following property? For every continuous function  $f(x)$  on the closed interval  $[0, 1]$  satisfying  $f(0) = f(1) = 0$ , there is some value of  $x$  for which  $f(x) = f(x + a)$ .
129. [2.5] Person  $A$  writes down one positive real number on each of two slips of paper, so that one of the numbers is twice the other. Person  $A$  then turns the slips face down and asks person  $B$  to choose a slip.  $A$  agrees to pay  $B$  the amount on the slip (in dollars) which  $B$  has chosen. After  $B$  picks his slip and turns it over, showing a number  $x$ ,  $A$  then says that  $B$  if he wants can choose the other slip instead.  $B$  reasons as follows: There is a 50% chance that the number on the other slip is  $2x$ , and a 50% chance it is  $x/2$ . Therefore the expected amount of money  $B$  will receive if he chooses the other slip is  $.5(2x) + .5(x/2) = 5x/4 > x$ . Thus  $B$  should always choose the other slip. Is there a flaw in this reasoning?
130. [3.5] Let  $x$  be a positive real number. Show that a square can be tiled with finitely many rectangles (in any orientation) similar to a  $1 \times x$  rectangle if and only if  $x$  is algebraic (satisfies a polynomial equation with integer coefficients) and all its conjugates (the other roots of the polynomial of least degree satisfied by  $x$ ) have positive real part. For instance,  $x = \sqrt{2}$  and  $x = 1 + \sqrt{2}$  cannot be done, since the conjugates are  $-\sqrt{2} < 0$  and  $1 - \sqrt{2} < 0$ . However,  $x = \frac{3}{2} + \sqrt{2}$  is possible since  $\frac{3}{2} - \sqrt{2} > 0$ .

131. [3] Let  $F(x) = (1 - x)(1 - x^2)(1 - x^3)(1 - x^5)(1 - x^8) \cdots$ , where the exponents are Fibonacci numbers. Show that every coefficient of  $F(x)$  (when expanded as a power series in  $x$ ) is 0, 1, or  $-1$ .
132. [3] Suppose that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a function such that  $f^2$  and  $f^3$  are infinitely differentiable on all of  $\mathbb{R}$ . Is the same true for  $f$ ?
133. [2.5] Let  $S$  be an  $n$ -element set and  $\mathcal{A}$  a collection of subsets of  $S$  such that (a) every  $T \in \mathcal{A}$  has an odd number of elements, and (b) the intersection of any two (distinct) subsets in  $\mathcal{A}$  has an even number of elements. Show that  $\#\mathcal{A} \leq n$ .