

PROBLEMS ON GENERATING FUNCTIONS

1. Let $f(m, 1) = f(1, n) = 1$ for $m \geq 1, n \geq 1$, and let

$$f(m, n) = f(m - 1, n) + f(m, n - 1) + f(m - 1, n - 1) \text{ for } m > 1 \text{ and } n > 1.$$

Also let

$$S(n) = \sum_{a+b=n} f(a, b), \quad a \geq 1 \text{ and } b \geq 1.$$

Prove that

$$S(n + 2) = S(n) + 2S(n + 1) \text{ for } n \geq 2.$$

2. Let $x^{(n)} = x(x - 1) \cdots (x - n + 1)$ for n a positive integer, and let $x^{(0)} = 1$. Prove that

$$(x + y)^{(n)} = \sum_{k=0}^n \binom{n}{k} x^{(k)} y^{(n-k)}.$$

3. For a set with n elements, how many subsets are there whose cardinality (the number of elements in the subset) is respectively $\equiv 0 \pmod{3}$, $\equiv 1 \pmod{3}$, $\equiv 2 \pmod{3}$? In other words, calculate

$$s_{i,n} = \sum_{k \equiv i \pmod{3}} \binom{n}{k} \text{ for } i = 0, 1, 2.$$

Your result should be strong enough to permit direct evaluation of the numbers $s_{i,n}$ and to show clearly the relationship of $s_{0,n}$ and $s_{1,n}$ and $s_{2,n}$ to each other for all positive integers n . In particular, show the relationships among these three sums for $n = 1000$. [An illustration of the definition of $s_{i,n}$ is $s_{0,6} = \binom{6}{0} + \binom{6}{3} + \binom{6}{6} = 22$.]

4. Given the power series

$$a_0 + a_1x + a_2x^2 + \cdots$$

in which

$$a_n = (n^2 + 1)3^n,$$

show that there is a relationship of the form

$$a_n + pa_{n+1} + qa_{n+2} + ra_{n+3} = 0,$$

in which p, q, r are constants independent of n . Find these constants and the sum of the power series.

5. Show that

$$x + \frac{2}{3}x^3 + \frac{2}{3} \frac{4}{5}x^5 + \frac{2}{3} \frac{4}{5} \frac{6}{7}x^7 + \cdots = \frac{\arcsin x}{\sqrt{1-x^2}}.$$

6. Let k be a positive integer and let $m = 6k - 1$. Let

$$S(m) = \sum_{j=1}^{2k-1} (-1)^{j+1} \binom{m}{3j-1}.$$

For example with $k = 3$,

$$S(17) = \binom{17}{2} - \binom{17}{5} + \binom{17}{8} - \binom{17}{11} + \binom{17}{14}.$$

Prove that $S(m)$ is never zero.

7. For nonnegative integers n and k , define $Q(n, k)$ to be the coefficient of x^k in the expansion of $(1 + x + x^2 + x^3)^n$. Prove that

$$Q(n, k) = \sum_{j=0}^n \binom{n}{j} \binom{n}{k-2j},$$

where $\binom{a}{b}$ is the standard binomial coefficient.

8. Let $a_{m,n}$ denote the coefficient of x^n in the expansion of $(1 + x + x^2)^m$. Prove that for all $k \geq 0$,

$$0 \leq \sum_{i=0}^{\lfloor 2k/3 \rfloor} (-1)^i a_{k-i,i} \leq 1.$$

9. Consider the power series expansion

$$\frac{1}{1 - 2x - x^2} = \sum_{n=0}^{\infty} a_n x^n.$$

Prove that, for each integer $n \geq 0$, there is an integer m such that

$$a_n^2 + a_{n+1}^2 = a_m.$$

10. Let $A = \{(x, y) : 0 \leq x, y \leq 1\}$. For $(x, y) \in A$, let

$$S(x, y) = \sum_{\frac{1}{2} \leq \frac{m}{n} \leq 2} x^m y^n,$$

where the sum ranges over all pairs (m, n) of positive integers satisfying the indicated inequalities. Evaluate

$$\lim_{(x,y) \rightarrow (1,1), (x,y) \in A} (1 - xy^2)(1 - x^2y)S(x, y).$$

11. For a set S of nonnegative integers, let $r_S(n)$ denote the number of ordered pairs (s_1, s_2) such that $s_1 \in S$, $s_2 \in S$, $s_1 \neq s_2$, and $s_1 + s_2 = n$. Is it possible to partition the nonnegative integers into two sets A and B in such a way that $r_A(n) = r_B(n)$ for all n ?
12. For positive integers m and n , let $f(m, n)$ denote the number of n -tuples (x_1, x_2, \dots, x_n) of integers such that $|x_1| + |x_2| + \dots + |x_n| \leq m$. Show that $f(m, n) = f(n, m)$.
13. Let S_n denote the set of all permutations of the numbers $1, 2, \dots, n$. For $\pi \in S_n$, let $\sigma(\pi) = 1$ if π is an even permutation and $\sigma(\pi) = -1$ if π is an odd permutation. Also, let $\nu(\pi)$ denote the number of fixed points of π . Show that

$$\sum_{\pi \in S_n} \frac{\sigma(\pi)}{\nu(\pi) + 1} = (-1)^{n+1} \frac{n}{n+1}.$$

14. Given $a_0 = 1$ and $a_{n+1} = (n+1)a_n - \binom{n}{2}a_{n-2}$ for $n \geq 0$, compute $y = \sum_{n \geq 0} a_n \frac{x^n}{n!}$.

15. Solve the recurrence

$$(n+1)(n+2)a_{n+2} - 3(n+1)a_{n+1} + 2a_n = 0,$$

with the initial conditions $a_0 = 2$, $a_1 = 3$.

16. Find the coefficients of the power series $y = 1 + 3x + 15x^2 + 184x^3 + 495x^4 + \cdots$ satisfying

$$(27x - 4)y^3 + 3y + 1 = 0.$$

17. Find the unique power series $y = 1 + \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \frac{1}{30240}x^6 + \cdots$ such that for all $n \geq 0$, the coefficient of x^n in y^{n+1} is equal to 1. (Give a simple expression for y , not for the coefficients of y .)

18. Find the unique power series $y = 1 + x - \frac{1}{2}x^2 + \frac{2}{3}x^3 + \cdots$ such that the constant term is 1, the coefficient of x is 1, and for all $n \geq 2$ the coefficient of x^n in y^n is 0. (Give a simple formula for the coefficients of y , not for y itself.)

19. Let $f(m, 0) = f(0, n) = 1$ and $f(m, n) = f(m - 1, n) + f(m, n - 1) + f(m - 1, n - 1)$ for $m, n > 0$. Show that

$$\sum_{n=0}^{\infty} f(n, n)x^n = \frac{1}{\sqrt{1 - 6x + x^2}}.$$

20. Suppose that \mathbb{Z} is written as a disjoint union of $n < \infty$ arithmetic progressions, with differences $d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$. Show that $d_1 = d_2$.

21. Solve the following equation for the power series $F(x, y) = \sum_{m, n \geq 0} a_{mn}x^m y^n$, where $a_{mn} \in \mathbb{R}$:

$$(xy^2 + x - y)F(x, y) = xF(x, 0) - y.$$

The point is to make sure that your solution has a power series expansion at $(0, 0)$.

22. Find a simple description of the coefficients $a_n \in \mathbb{Z}$ of the power series $F(x) = x + a_2x^2 + a_3x^3 + \cdots$ satisfying the functional equation

$$F(x) = (1 + x)F(x^2) + \frac{x}{1 - x^2}.$$

23. Let S be the set of sequences of length 2018 whose terms are in the set $\{1, 2, 3, 4, 5, 6, 10\}$ and sum to 3860. Prove that the cardinality of S is at most

$$2^{3860} \left(\frac{2018}{2048} \right)^{2018}.$$