PROBLEMS ON "HIDDEN" INDEPENDENCE AND UNIFORMITY

All the problems below when looked at the right way, can be solved by elegant arguments avoiding induction, recurrence relations, complicated sums, etc. They all have a vague theme in common, related to certain probabilities being either uniform or independent. However, it is not necessary to look at a problem from this point of view in order to find the elegant solution. If you solve a problem in a complicated way, the answer might suggest to you a simpler method.

1. Slips of paper with the numbers from 1 to 99 are placed in a hat. Five numbers are randomly drawn out of the hat one at a time (without replacement). What is the probability that the numbers are chosen in increasing order?

2. In how many ways can a positive integer n be written as a sum of positive integers, taking order into account? For instance, 4 can be written as a sum in the eight ways $4 = 3 + 1 = 1 + 3 = 2 + 2 = 2 + 1 + 1 = 1 + 2 + 1 = 1 + 1 + 2 = 1 + 1 + 1 + 1$.

3. How many 8×8 matrices of 0's and 1's are there, such that every row and column contains an odd number of 1's?

4. Let $f(n)$ be the number of ways to take an n-element set S, and, if S has more than one element, to partition S into two disjoint nonempty subsets S_1 and S_2, then to take one of the sets S_1, S_2 with more than one element and partition it into two disjoint nonempty subsets S_3 and S_4, then to take one of the sets with more than one element not yet partitioned and partition it into two disjoint nonempty subsets, etc., always taking a set with more than one element that is not yet partitioned and partitioning it into two nonempty disjoint subsets, until only one-element subsets remain. For example, we could start with 12345678 (short for $\{1, 2, 3, 4, 5, 6, 7, 8\}$), then partition it into 126 and 34578, then partition 34578 into 4 and 3578, then 126 into 6 and 12, then 3578 into 37 and 58, then 58 into 5 and 8, then 12 into 1 and 2, and finally 37 into 3 and 7. (The order we partition the sets is important; for instance, partitioning 1234 into 12 and 34, then 12 into 1 and 2, and then 34 into 3 and 4, is different from partitioning 1234 into 12 and 34, then 34 into 3 and 4, and then 12 into 1 and 2. However, partitioning 1234 into 12 and 34 is the same as partitioning it into 34 and 12.) Find a simple formula for $f(n)$. For instance, $f(1) = 1$, $f(2) = 1$, $f(3) = 3$, and $f(4) = 18$.

5. Fix positive integers n and k. Find the number of k-tuples $(S_1, S_2, \ldots, S_k)$ of subsets S_i of $\{1, 2, \ldots, n\}$ subject to each of the following conditions:
   (a) $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k$
   (b) The S_i's are pairwise disjoint.
   (c) $S_1 \cap S_2 \cap \cdots \cap S_k = \emptyset$
   (d) $S_1 \subseteq S_2 \supseteq S_3 \subseteq S_4 \supseteq S_5 \subseteq \cdots S_k$ (The symbols $\subseteq$ and $\supseteq$ alternate.)

6. Let p be a prime number and $1 \leq k \leq p-1$. How many k-element subsets $\{a_1, \ldots, a_k\}$ of $\{1, 2 \ldots, p\}$ are there such that $a_1 + \cdots + a_k \equiv 0 \pmod{p}$?

7. Let π be a random permutation of $1, 2, \dots, n$. Fix a positive integer $1 \le k \le n$. What is the probability that in the disjoint cycle decomposition of π, the length of the cycle containing 1 is k? In other words, what is the probability that k is the least positive integer for which $\pi^k(1) = 1$?

8. Let π be a random permutation of $1, 2, \dots, n$. What is the probability that 1 and 2 are in the same cycle of π?

9. Choose n real numbers $x_1, \dots, x_n$ uniformly and independently from the interval $[0, 1]$. What is the expected value of $\min_i x_i$, the minimum of $x_1, \dots, x_n$?

10. (a) Let m and n be nonnegative integers. Evaluate the integral

$$B(m, n) = \int_0^1 x^m (1-x)^n dx,$$

by interpreting the integral as a probability.

(b) Let R be the region consisting of all triples (x, y, z) of nonnegative real numbers satisfying $x + y + z \le 1$. Let $w = 1 - x - y - z$. Express the value of the triple integral (taken over the region R)

$$\iiint x^1 y^9 z^8 w^4 \, dx \, dy \, dz$$

in the form $a!\, b!\, c!\, d!/n!$, where $a,\ b,\ c,\ d$, and n are positive integers.

11. (a) Choose n points at random (uniformly and independently) on the circumference of a circle. Find the probability p_n that all the points lie on a semicircle. (For instance, $p_1 = p_2 = 1$.)

(b) Fix $\theta < 2\pi$ and find the probability that the n points lie on an arc subtending an angle θ .

(c) Choose four points at random on the surface of a sphere. Find the probability that the center of the sphere is contained within the convex hull of the four points.

(d) (more difficult—I don't know an elegant proof) Choose n points uniformly at random in a square (or more generally, a parallelogram). Show that the probability that the points are in convex position (i.e., each is a vertex of their convex hull) is given by

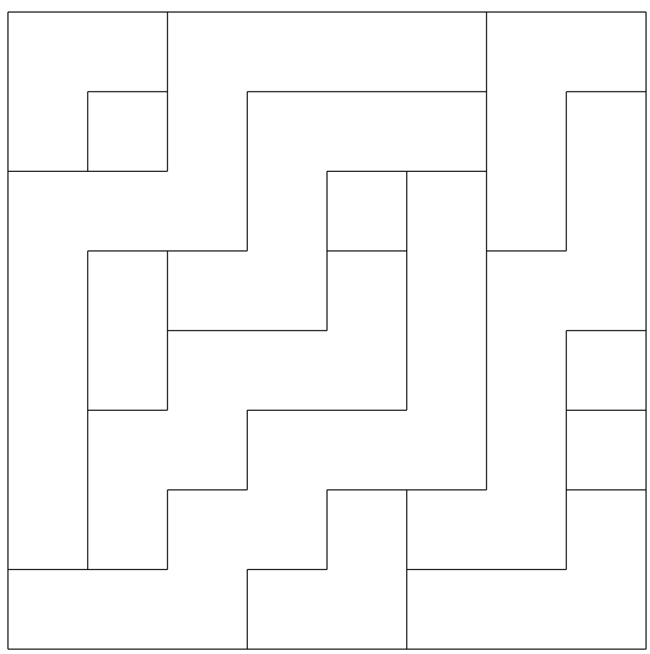
$$P_n = \left[\frac{1}{n!} \binom{2n-2}{n-1} \right]^2 .$$

12. Passengers $P_1, \dots, P_n$ enter a plane with n seats. Each passenger has a different assigned seat. The first passenger sits in the wrong seat. Thereafter, each passenger either sits in their seat if unoccupied or otherwise sits in a random unoccupied seat. What is the probability that the last passenger sits in his or her own seat?

13. Let $x_1, x_2, \ldots, x_n$ be n points (in that order) on the circumference of a circle. A person starts at the point x_1 and walks to one of the two neighboring points with probability $1/2$ for each. The person continues to walk in this way, always moving from the present point to one of the two neighboring points with probability $1/2$ for each. Find the probability p_i that the point x_i is the last of the n points to be visited for the first time. In other words, find the probability that when x_i is visited for the first time, all the other points will have already been visited. For instance, $p_1 = 0$ (when $n > 1$), since x_1 is the *first* of the n points to be visited.

14. There are n parking spaces $1, 2, \ldots, n$ (in that order) on a one-way street. Cars $C_1, \ldots, C_n$ enter the street in that order and try to park. Each car C_i has a preferred space a_i. A car will drive to its preferred space and try to park there. If the space is already occupied, the car will park in the next available space. If the car must leave the street without parking, then the process fails. If $\alpha = (a_1, \ldots, a_n)$ is a sequence of preferences that allows every car to park, then we call α a *parking function*. For instance, there are 16 parking functions of length 3, given by (abbreviating $(1, 1, 1)$ as 111, etc.) 111, 112, 121, 211, 113, 131, 311, 122, 212, 221, 123, 132, 213, 231, 312, 321. Show that the number of parking functions of length n is equal to $(n+1)^{n-1}$.

15. A *snake* on the 8×8 chessboard is a nonempty subset S of the squares of the board obtained as follows: Start at one of the squares and continue walking one step up or to the right, stopping at any time. The squares visited are the squares of the snake. Here is an example of the 8×8 chessboard covered with disjoint snakes.



Find the total number of ways to cover an 8×8 chessboard with disjoint snakes. Generalize to an $m \times n$ chessboard.

16. Let n balls be thrown uniformly and independently into n bins.

(a) Find the probability that bin 1 is empty.

(b) Find the expected number of empty bins.

17. Consider $2m$ persons forming m couples who live together at a given time. Suppose that at some later time, the probability of each person being alive is p, independently of other persons. At that later time, let A be the number of persons that are alive and let S be the number of couples in which both partners are alive. For any number of total surviving persons a, find expected number of surviving couples, i.e., $\mathbb{E}[S|A = a]$.

18. Let X_1, X_2 and X_3 be three independent, continuous random variables with the same distribution. Given that X_2 is smaller than X_3, what is the conditional probability that X_1 is smaller than X_2?

19. In game show each contestant i spins an infinitely calibrated "fair" wheel of fortune, which assigns the contestant a real number X_i between 1 and 100.

    (a) Find $\mathbb{P}(X_1 < X_2)$. Explain your answer.

    (b) Find $\mathbb{P}(X_1 < X_2, X_1 < X_3)$, i.e., find the probability that the first contestant will have the smallest value of the first three contestants.

    (c) Consider a new random variable N, which is integer valued. N is the index of the first contestant who is assigned a smaller number than contestant 1. As an illustration, if contestant 1 has a smaller value than contestants 2, 3, and 4, but contestant 5 has a smaller value than contestant 1 ($X_5 < X_1$), then $N = 5$. Find $\mathbb{P}(N > n)$ as a function of n.

20. The adventures of Ant Alice. On a meter-long rod sit 25 ants placed uniformly at random, each facing a uniformly chosen direction (east or west). They proceed to march forward at 1 cm/sec; whenever two ants collide, they reverse directions. Ants fall off the rod when they reach either endpoint. Alice is the ant initially positioned 13th from the west.

    (a) How long does it take before we can be certain that Alice is off the rod?

    (b) What is the probability that Alice falls off the rod facing the same direction as her initial direction?

    (c) What is the probability that Alice is the last ant to fall off the rod?

    (d) What is the expected number of collisions that Alice has?

    (e) What is the probability that Alice has more collisions than any other ant?

    (f) Alice has a cold, which is transmitted from ant to ant instantly upon collision. How is the expected number of ants infected in the process?