PROBLEMS ON GENERATING FUNCTIONS

Note. All the problems below can be done using generating functions. Many of them can also be done by other methods. However, you should hand in only solutions which use generating functions. **No credit** for solving a problem without using generating functions!

1. Let $f(m,1) = f(1,n) = 1$ for $m \geq 1$, $n \geq 1$, and let

   $f(m,n) = f(m-1,n) + f(m,n-1) + f(m-1,n-1)$ for $m > 1$ and $n > 1$.

   Also let
   $$S(n) = \sum_{a+b=n} f(a,b),\ \ a \geq 1 \text{ and } b \geq 1.$$
   Prove that
   $$S(n+2) = S(n) + 2S(n+1) \text{ for } n \geq 2.$$

2. Let $x^{(n)} = x(x-1)\cdots(x-n+1)$ for n a positive integer, and let $x^{(0)} = 1$. Prove that
   $$(x+y)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} x^{(k)} y^{(n-k)}.$$

   Note: $\displaystyle\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{1 \cdot 2 \cdots k}.$

3. For a set with n elements, how many subsets are there whose cardinality (the number of elements in the subset) is respectively $\equiv 0 (\text{mod } 3)$, $\equiv 1 (\text{mod } 3)$, $\equiv 2 (\text{mod } 3)$? In other words, calculate
   $$s_{i,n} = \sum_{k \equiv i \pmod 3} \binom{n}{k} \text{ for } i = 0, 1, 2.$$

   Your result should be strong enough to permit direct evaluation of the numbers $s_{i,n}$ and to show clearly the relationship of $s_{0,n}$ and $s_{1,n}$ and $s_{2,n}$ to each other for all positive integers n. In particular, show the relationships among these three sums for $n = 1000$. [An illustration of the definition of $s_{i,n}$ is $s_{0,6} = \binom{6}{0} + \binom{6}{3} + \binom{6}{6} = 22$.]

4. Given the power series

$$a_0 + a_1 x + a_2 x^2 + \cdots$$

in which

$$a_n = (n^2 + 1)3^n,$$

show that there is a relationship of the form

$$a_n + p a_{n+1} + q a_{n+2} + r a_{n+3} = 0,$$

in which $p,\ q,\ r$ are constants independent of n. Find these constants and the sum of the power series.

5. Show that

$$x + \frac{2}{3}x^3 + \frac{2}{3}\frac{4}{5}x^5 + \frac{2}{3}\frac{4}{5}\frac{6}{7}x^7 + \cdots = \frac{\text{arc sin } x}{\sqrt{1-x^2}}.$$

NOTE (not on Putnam Exam): $\text{arc sin}\, x$ is the same as $\sin^{-1} x$.

6. Let k be a positive integer and let $m = 6k - 1$. Let

$$S(m) = \sum_{j=1}^{2k-1} (-1)^{j+1} \binom{m}{3j-1}.$$

For example with $k = 3$,

$$S(17) = \binom{17}{2} - \binom{17}{5} + \binom{17}{8} - \binom{17}{11} + \binom{17}{14}.$$

Prove that $S(m)$ is never zero. [As usual, $\binom{m}{r} = \frac{m!}{r!(m-r)!}$.]

7. For nonnegative integers n and k, define $Q(n,k)$ to be the coefficient of x^k in the expansion of $(1 + x + x^2 + x^3)^n$. Prove that

$$Q(n,k) = \sum_{j=0}^{n} \binom{n}{j}\binom{n}{k-2j},$$

where $\binom{a}{b}$ is the standard binomial coefficient. (Reminder: For integers a and b with $a \geq 0$, $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ for $0 \leq b \leq a$, and $\binom{a}{b} = 0$ otherwise.)

8. Let $a_{m,n}$ denote the coefficient of x^n in the expansion of $(1+x+x^2)^m$. Prove that for all $k \geq 0$,

$$0 \leq \sum_{i=0}^{\lfloor 2k/3 \rfloor} (-1)^i a_{k-i,i} \leq 1.$$

9. Consider the power series expansion

$$\frac{1}{1-2x-x^2} = \sum_{n=0}^{\infty} a_n x^n.$$

Prove that, for each integer $n \geq 0$, there is an integer m such that

$$a_n^2 + a_{n+1}^2 = a_m.$$

10. Let $A = \{(x,y) : 0 \leq x, y \leq 1\}$. For $(x,y) \in A$, let

$$S(x,y) = \sum_{\frac{1}{2} \leq \frac{m}{n} \leq 2} x^m y^n,$$

where the sum ranges over all pairs (m,n) of positive integers satisfying the indicated inequalities. Evaluate

$$\lim_{(x,y)\to(1,1),\ (x,y)\in A} (1-xy^2)(1-x^2y)S(x,y).$$

11. For a set S of nonnegative integers, let $r_S(n)$ denote the number of ordered pairs (s_1, s_2) such that $s_1 \in S$, $s_2 \in S$, $s_1 \neq s_2$, and $s_1 + s_2 = n$. Is it possible to partition the nonnegative integers into two sets A and B in such a way that $r_A(n) = r_B(n)$ for all n?

12. For positive integers m and n, let $f(m,n)$ denote the number of n-tuples $(x_1, x_2, \ldots, x_n)$ of integers such that $|x_1| + |x_2| + \cdots + |x_n| \leq m$. Show that $f(m,n) = f(n,m)$.

13. Let S_n denote the set of all permutations of the numbers $1, 2, \ldots, n$. For $\pi \in S_n$, let $\sigma(\pi) = 1$ if π is an even permutation and $\sigma(\pi) = -1$ if π is an odd permutation. Also, let $\nu(\pi)$ denote the number of fixed points of π. Show that

$$\sum_{\pi \in S_n} \frac{\sigma(\pi)}{\nu(\pi)+1} = (-1)^{n+1} \frac{n}{n+1}.$$

14. Given $a_0 = 1$ and $a_{n+1} = (n+1)a_n - \binom{n}{2}a_{n-2}$ for $n \geq 0$, compute $y = \sum_{n \geq 0} a_n \frac{x^n}{n!}$.

15. Solve the recurrence

$$(n+1)(n+2)a_{n+2} - 3(n+1)a_{n+1} + 2a_n = 0,$$

with the initial conditions $a_0 = 2,\ a_1 = 3$.

16. Find the coefficients of the power series $y = 1 + 3x + 15x^2 + 184x^3 + 495x^4 + \cdots$ satisfying

$$(27x - 4)y^3 + 3y + 1 = 0.$$

17. Find the unique power series $y = 1 + x - \frac{1}{2}x^2 + \frac{2}{3}x^3 + \cdots$ such that the constant term is 1, the coefficient of x is 1, and for all $n \geq 2$ the coefficient of x^n in y^n is 0. (Give a simple formula for the coefficients of y, not for y itself.)

18. Let $f(m,0) = f(0,n) = 1$ and $f(m,n) = f(m-1,n) + f(m,n-1) + f(m-1,n-1)$ for $m,n > 0$. Show that

$$\sum_{n=0}^{\infty} f(n,n)x^n = \frac{1}{\sqrt{1 - 6x + x^2}}.$$

19. Suppose that $\mathbb{Z}$ is written as a disjoint union of $n < \infty$ arithmetic progressions, with differences $d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$. Show that $d_1 = d_2$.

20. Solve the following equation for the power series $F(x,y) = \sum_{m,n \geq 0} a_{mn}x^m y^n$, where $a_{mn} \in \mathbb{R}$:

$$(xy^2 + x - y)F(x,y) = xF(x,0) - y.$$

The point is to make sure that your solution has a power series expansion at $(0,0)$.

21. Find a simple description of the coefficients $a_n \in \mathbb{Z}$ of the power series $F(x) = x + a_2x^2 + a_3x^3 + \cdots$ satisfying the functional equation

$$F(x) = (1+x)F(x^2) + \frac{x}{1-x^2}.$$