CONGRUENCES AND DIVISIBILITY

1. Let $n_1, n_2, \ldots, n_s$ be distinct integers such that

$$(n_1 + k)(n_2 + k) \cdots (n_s + k)$$

is an integral multiple of $n_1 n_2 \cdots n_s$ for every integer k. For each of the following assertions, give a proof or a counterexample:

- $|n_i| = 1$ for some i.
- If further all n_i are positive, then

$$\{n_1, n_2, \ldots, n_s\} = \{1, 2, \ldots, s\}.$$

2. How many coefficients of the polynomial

$$P_n(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n} (x_i + x_j)$$

are odd?

3. If p is a prime number greater than 3 and $k = \lfloor 2p/3 \rfloor$, prove that the sum

$$\binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{k}$$

of binomial coefficients is divisible by p^2.

4. Do there exist positive integers a and b with $b - a > 1$ such for every $a < k < b$, either $\gcd(a, k) > 1$ or $\gcd(b, k) > 1$?

5. Suppose that $f(x)$ and $g(x)$ are polynomials (with $f(x)$ not identically 0) taking integers to integers such that for all $n \in \mathbb{Z}$, either $f(n) = 0$ or $f(n) | g(n)$. Show that $f(x) | g(x)$, i.e., there is a polynomial $h(x)$ with rational coefficients such that $g(x) = f(x)h(x)$.

6. Let q be an odd positive integer, and let N_q denote the number of integers a such that $0 < a < q/4$ and $\gcd(a, q) = 1$. Show that N_q is odd if and only if q is of the form p^k with k a positive integer and p a prime congruent to 5 or 7 modulo 8.

7. Let p be in the set $\{3, 5, 7, 11, \ldots\}$ of odd primes, and let

$$F(n) = 1 + 2n + 3n^2 + \cdots + (p-1)n^{p-2}.$$

Prove that if a and b are distinct integers in $\{0, 1, 2, \ldots, p-1\}$ then $F(a)$ and $F(b)$ are not congruent modulo p, that is, $F(a) - F(b)$ is not exactly divisible by p.

8. Do there exist 1,000,000 consecutive integers each of which contains a repeated prime factor?

♡ 9. A positive integer n is *powerful* if for every prime p dividing n, we have that p^2 divides n. Show that for any $k \geq 1$ there exist k consecutive integers, none of which is powerful.

10. Show that for any $k \geq 1$ there exist k consecutive positive integers, none of which is a sum of two squares. (You may use the fact that a positive integer n is a sum of two squares if and only if for every prime $p \equiv 3 \pmod 4$, the largest power of p dividing n is an even power of p.)

11. Prove that every positive integer has a multiple whose decimal representation involves all ten digits.

12. Prove that among any ten consecutive integers at least one is relatively prime to each of the others.

13. Find the length of the longest sequence of equal nonzero digits in which an integral square can terminate (in base 10), and find the smallest square which terminates in such a sequence.

14. Show that if n is an odd integer greater than 1, then n does not divide $2^n + 2$.

15. * For positive integer a, we define the series

$$f_a(q) = \sum_{k \geq 0, ak+1 \text{ is a square}} q^k.$$

Find all positive integer triples (a, b, c) such that

$$f_a(q) \equiv f_b(q) f_c(q) \mod 2$$

which means that the corresponding coefficients match modulo 2. (**Hint**: Use a computer to find a few triple, then look for patterns.)

16. Define a sequence $\{a_i\}$ by $a_1 = 3$ and $a_{i+1} = 3^{a_i}$ for $i \geq 1$. Which integers between 00 and 99 inclusive occur as the last two digits in the decimal expansion of infinitely many a_i?

17. What is the units (i.e., rightmost) digit of

$$\left[\frac{10^{20000}}{10^{100} + 3} \right]?$$

Here $[x]$ is the greatest integer $\leq x$.

18. Suppose p is an odd prime. Prove that

$$\sum_{j=0}^{p} \binom{p}{j} \binom{p+j}{j} \equiv 2^p + 1 \pmod{p^2}.$$

19. Prove that for $n \geq 2$,

$$\overbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}^{n \text{ terms}} \equiv \overbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}^{n-1 \text{ terms}} \pmod{n}.$$

20. The sequence $(a_n)_{n \geq 1}$ is defined by $a_1 = 1$, $a_2 = 2$, $a_3 = 24$, and, for $n \geq 4$,

$$a_n = \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}.$$

Show that, for all n, a_n is an integer multiple of n.

♡ 21. Prove that the expression
$$\frac{\gcd(m,n)}{n}\binom{n}{m}$$
is an integer for all pairs of integers $n \geq m \geq 1$.

22. Show that for each positive integer n,
$$n! = \prod_{i=1}^{n} \operatorname{lcm}\{1,2,\dots,\lfloor n/i \rfloor\}.$$
(Here lcm denotes the least common multiple, and $\lfloor x \rfloor$ denotes the greatest integer $\leq x$.)

23. Define a sequence $\{u_n\}_{n=0}^{\infty}$ by $u_0 = u_1 = u_2 = 1$, and thereafter by the condition that
$$\det\begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!$$
for all $n \geq 0$. Show that u_n is an integer for all n. (By convention, $0! = 1$.)

♡ 24. Let p be a prime number. Let $h(x)$ be a polynomial with integer coefficients such that $h(0), h(1), \dots, h(p^2-1)$ are distinct modulo p^2. Show that $h(0), h(1), \dots, h(p^3-1)$ are distinct modulo p^3.

25. Let $f(x) = a_0 + a_1 x + \cdots$ be a power series with integer coefficients, with $a_0 \neq 0$. Suppose that the power series expansion of $f'(x)/f(x)$ at $x = 0$ also has integer coefficients. Prove or disprove that $a_0 | a_n$ for all $n \geq 0$.

26. Let S be a set of rational numbers such that

(a) $0 \in S$;
(b) If $x \in S$ then $x+1 \in S$ and $x-1 \in S$; and
(c) If $x \in S$ and $x \notin \{0,1\}$, then $1/(x(x-1)) \in S$.

Must S contain all rational numbers?

♡ 27. Prove that for each positive integer n, the number $10^{10^{10^n}} + 10^{10^n} + 10^n - 1$ is not prime.

28. Let p be an odd prime. Show that for at least $(p+1)/2$ values of n in $\{0,1,2,\dots,p-1\}$, $\sum_{k=0}^{p-1} k!n^k$ is not divisible by p.

29. Let a and b be distinct rational numbers such that $a^n - b^n$ is an integer for all positive integers n. Prove or disprove that a and b must themselves be integers.

30. Find the smallest integer $n \geq 2$ for which there exists an integer m with the following property: for each $i \in \{1,\dots,n\}$, there exists $j \in \{1,\dots,n\}$ different from i such that $\gcd(m+i, m+j) > 1$.

31. Let p be an odd prime number such that $p \equiv 2 \pmod 3$. Define a permutation π of the residue classes modulo p by $\pi(x) \equiv x^3 \pmod p$. Show that π is an even permutation if and only if $p \equiv 3 \pmod 4$.

32. Suppose that a positive integer N can be expressed as the sum of k consecutive positive integers

$$N = a + (a+1) + (a+2) + \cdots + (a+k-1)$$

for $k = 2017$ but for no other values of $k > 1$. Considering all positive integers N with this property, what is the smallest positive integer a that occurs in any of these expressions?

33. Let n be a positive integers. Prove that

$$\sum_{k=1}^{n} (-1)^{\lfloor k(\sqrt{2}-1) \rfloor} \geq 0.$$

34. How many positive integers N satisfy all of the following three conditions? (i) N is divisible by 2020. (ii) N has at most 2020 decimal digits. (iii) The decimal digits of N are a string of consecutive ones followed by a string of consecutive zeros.

35. Find all positive integers $n < 10^{100}$ for which simultaneously n divides 2^n, $n-1$ divides $2^n - 1$, and $n-2$ divides $2^n - 2$.

36. Define the Fibonacci numbers by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Let k be a positive integer. Suppose that for every positive integer m there exists a positive integer n such that $m \mid F_n - k$. Must k be a Fibonacci number?

37. Suppose that

$$f(x) = \sum_{i=0}^{\infty} c_i x^i$$

is a power series for which each coefficient c_i is 0 or 1. Show that if $f(2/3) = 3/2$, then $f(1/2)$ must be irrational.

38. Let $P(x)$ be a polynomial with integer coefficients such that n divides $P(2^n)$ for all positive integers n. Prove that P is the zero polynomial.

♡ 39. Let $a_1, a_2, \ldots, a_n$ be positive integers with product P, where n is an odd positive integer. Prove that

$$\gcd(a_1^n + P, a_2^n + P, \ldots, a_n^n + P) \leq 2 \gcd(a_1, \ldots, a_n)^n.$$

40. Let $\mathbb{N}$ denote the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$n + f(m) \mid f(n) + nf(m)$$

for all $m, n \in \mathbb{N}$.

41. Determine all positive integers M such that the sequence $a_0, a_1, a_2, \cdots$ defined by

$$a_0 = M + \frac{1}{2} \qquad \text{and} \qquad a_{k+1} = a_k \lfloor a_k \rfloor \quad \text{for } k = 0, 1, 2, \cdots$$

contains at least one integer term.

42. Find all positive integers $n > 2$ such that

$$n! \mid \prod_{p < q \leq n, p, q \text{ primes}} (p+q).$$

43. Let $a_0 = 1, a_1 = 2$, and $a_n = 4a_{n-1} - a_{n-2}$ for $n \geq 2$. Find an odd prime factor of a_{2015}.