ANALYSIS PROBLEMS

We use the notation $f(x) \sim g(x)$ to mean $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$. One says that $f(x)$ is *asymptotic* to $g(x)$.

Exercises.
These are problems whose techniques are worth knowing, but will generally only form components of larger solutions. (Submit at most one.)

1. Show that $\int_0^\infty \frac{\cos(ax)}{1+x^2}dx$ exists for $a \in \mathbb{R}$ and compute its value.
2. Find a simple function $f(x)$ for which $x^{1/x} - 1 \sim f(x)$ as $x \to \infty$.
3. For what pairs (a, b) of positive real numbers does the improper integral
$$\int_b^\infty \left(\sqrt{\sqrt{x+a}-\sqrt{x}} - \sqrt{\sqrt{x}-\sqrt{x-b}}\right) dx$$
converge?
4. Let a_n be the unique positive root of $x^n + x = 1$. Find a simple function $f(n)$ for which $1 - a_n \sim f(n)$ as $n \to \infty$.
5. For each continuous function $f : [0, 1] \to \mathbb{R}$, let $I(f) = \int_0^1 x^2 f(x)dx$ and $J(f) = \int_0^1 x f(x)^2 dx$. Find the maximum value of $I(f) - J(f)$ over all such functions f.

Problems.

6. For a positive real number a, calculate $\int_0^\infty t^{-1/2} e^{-a(t+t^{-1})} dt$.
7. Let f be a function on $[0, \infty)$, differentiable and satisfying
$$f'(x) = -3f(x) + 6f(2x)$$
for $x > 0$. Assume that $|f(x)| \le e^{-\sqrt{x}}$ for $x \ge 0$ (so that $f(x)$ tends rapidly to ∞ as x increases). For n a nonnegative integer, define
$$\mu_n = \int_0^\infty x^n f(x)dx$$
(the nth moment of f).
   (a) Express μ_n in terms of μ_0.
   (b) Prove that the sequence $\{\mu_n \cdot 3^n/n!\}$ always converges, and that the limit is 0 only if $\mu_0 = 0$.

♡ 8. Suppose f and g are non-constant, differentiable, real-valued functions defined on $(-\infty, \infty)$. Furthermore, suppose that for each pair of real numbers x and y,
$$\begin{aligned} f(x+y) &= f(x)f(y) - g(x)g(y), \\ g(x+y) &= f(x)g(y) + g(x)f(y). \end{aligned}$$
If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all x.

9. Let a and b be positive numbers. Find the largest number c, in terms of a and b, such that

$$a^x b^{1-x} \leq a\frac{\sinh ux}{\sinh u} + b\frac{\sinh u(1-x)}{\sinh u}$$

for all u with $0 < |u| \leq c$ and for all x, $0 < x < 1$. (Note: $\sinh u = (e^u - e^{-u})/2$.)

10. The function $K(x, y)$ is positive and continuous for $0 \leq x \leq 1, 0 \leq y \leq 1$, and the functions $f(x)$ and $g(x)$ are positive and continuous for $0 \leq x \leq 1$. Suppose that for all x, $0 \leq x \leq 1$,

$$\int_0^1 f(y)K(x, y)\, dy = g(x)$$

and

$$\int_0^1 g(y)K(x, y)\, dy = f(x).$$

Show that $f(x) = g(x)$ for $0 \leq x \leq 1$.

11. Evaluate

$$\int_0^\infty \left(x - \frac{x^3}{2} + \frac{x^5}{2 \cdot 4} - \frac{x^7}{2 \cdot 4 \cdot 6} + \cdots\right)\left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots\right) dx.$$

12. Let f be a twice-differentiable real-valued function satisfying

$$f(x) + f''(x) = -xg(x)f'(x),$$

where $g(x) \geq 0$ for all real x. Prove that $|f(x)|$ is bounded.

13. Prove that there is a constant C such that, if $p(x)$ is a polynomial of degree 1999, then

$$|p(0)| \leq C \int_{-1}^1 |p(x)|\, dx.$$

14. Find a real number c and a positive number L for which

$$\lim_{r \to \infty} \frac{r^c \int_0^{\pi/2} x^r \sin x dx}{\int_0^{\pi/2} x^r \cos x dx} = L.$$

15. Let $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$ be the vertices of a convex polygon which contains the origin in its interior. Prove that there exist positive real numbers x and y such that

$$(a_1, b_1)x^{a_1}y^{b_1} + (a_2, b_2)x^{a_2}y^{b_2} + \cdots + (a_n, b_n)x^{a_n}y^{b_n} = (0, 0).$$

16. Show that all solutions of the differential equation $y'' + e^x y = 0$ remain bounded as $x \to \infty$.

17. Let f be a real-valued function having partial derivatives and which is defined for $x^2 + y^2 \leq 1$ and is such that $|f(x, y)| \leq 1$. Show that there exists a point (x_0, y_0) in the interior of the unit circle for which

$$\left(\frac{\partial f}{\partial x}(x_0, y_0)\right)^2 + \left(\frac{\partial f}{\partial y}(x_0, y_0)\right)^2 \leq 16.$$

18. (a) On $[0,1]$, let f have a continuous derivative satisfying $0 < f'(x) \leq 1$. Also, suppose that $f(0) = 0$. Prove that

$$\left(\int_0^1 f(x)dx\right)^2 \geq \int_0^1 f(x)^3 dx.$$

(b) Find an example where equality occurs.

19. Let $P(t)$ be a nonconstant polynomial with real coefficients. Prove that the system of simultaneous equations

$$0 = \int_0^x P(t)\sin t dt = \int_0^x P(t)\cos t dt$$

has only finitely many real solutions x.

20. Let C be the class of all real valued continuously differentiable functions f on the interval $0 \leq x \leq 1$ with $f(0) = 0$ and $f(1) = 1$. Determine the largest real number u such that

$$u \leq \int_0^1 |f'(x) - f(x)| dx$$

for all $f \in C$.

21. Given a convergent series $\sum a_n$ of positive terms, prove that the series $\sum \sqrt[n]{a_1 a_2 \cdots a_n}$ must also be convergent.

22. Given that $f(x) + f'(x) \to 0$ as $x \to \infty$, prove that both $f(x) \to 0$ and $f'(x) \to 0$.

23. Suppose that $f''(x)$ is continuous on $\mathbb{R}$, and that $|f(x)| \leq a$ on $\mathbb{R}$, and $|f''(x)| \leq b$ on $\mathbb{R}$. Find the best possible bound $|f'(x)| \leq c$ on $\mathbb{R}$.

24. Let f be a real function with a continuous third derivative such that $f(x), f'(x), f''(x), f'''(x)$ are positive for all x. Suppose that $f'''(x) \leq f(x)$ for all x. Show that $f'(x) < 2f(x)$ for all x. (Note that we cannot replace 2 by 1 because of the function $f(x) = e^x$.)

25. Show that the improper integral

$$\lim_{B\to\infty} \int_0^B \sin(x)\, \sin(x^2)\, dx$$

converges.

♡ 26. Fix an integer $b \geq 2$. Let $f(1) = 1$, $f(2) = 2$, and for each $n \geq 3$, define $f(n) = nf(d)$, where d is the number of base-b digits of n. For which values of b does

$$\sum_{n=1}^{\infty} \frac{1}{f(n)}$$

converge?

27. Evaluate

$$\lim_{x\to 1^-} \prod_{n=0}^{\infty} \left(\frac{1+x^{n+1}}{1+x^n}\right)^{x^n}.$$

28. Find all differentiable functions $f:(0,\infty)\to(0,\infty)$ for which there is a positive real number a such that

$$f'\left(\frac{a}{x}\right)=\frac{x}{f(x)}$$

for all $x>0$.

29. Let k be an integer greater than 1. Suppose $a_0>0$, and define

$$a_{n+1}=a_n+\frac{1}{\sqrt[k]{a_n}}$$

for $n>0$. Evaluate

$$\lim_{n\to\infty}\frac{a_n^{k+1}}{n^k}.$$

30. Define $f\colon\mathbb{R}\to\mathbb{R}$ by

$$f(x)=\begin{cases}x & \text{if } x\le e\\ xf(\ln x) & \text{if } x>e.\end{cases}$$

Does $\sum_{n=1}^{\infty}\frac{1}{f(n)}$ converge?

31. Find all continuously differentiable functions $f:\mathbb{R}\to\mathbb{R}$ such that for every rational number q, the number $f(q)$ is rational and has the same denominator as q. (The denominator of a rational number q is the unique positive integer b such that $q=a/b$ for some integer a with $\gcd(a,b)=1$.) (Note: gcd means greatest common divisor.)

♡ 32. Functions f,g,h are differentiable on some open interval around 0 and satisfy the equations and initial conditions

$$f'=2f^2gh+\frac{1}{gh},\quad f(0)=1,$$
$$g'=fg^2h+\frac{4}{fh},\quad g(0)=1,$$
$$h'=3fgh^2+\frac{1}{fg},\quad h(0)=1.$$

Find an explicit formula for $f(x)$, valid in some open interval around 0.

33. Let $f:[0,1]^2\to\mathbb{R}$ be a continuous function on the closed unit square such that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous on the interior $(0,1)^2$. Let $a=\int_0^1 f(0,y)\,dy$, $b=\int_0^1 f(1,y)\,dy$, $c=\int_0^1 f(x,0)\,dx$, $d=\int_0^1 f(x,1)\,dx$. Prove or disprove: There must be a point (x_0,y_0) in $(0,1)^2$ such that

$$\frac{\partial f}{\partial x}(x_0,y_0)=b-a\quad\text{and}\quad\frac{\partial f}{\partial y}(x_0,y_0)=d-c.$$

34. Let $f:(1,\infty)\to\mathbb{R}$ be a differentiable function such that

$$f'(x)=\frac{x^2-(f(x))^2}{x^2((f(x))^2+1)}\qquad\text{for all } x>1.$$

Prove that $\lim_{x\to\infty}f(x)=\infty$.

♡ 35. Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = \frac{f(x+n) - f(x)}{n}$$

for all real numbers x and all positive integers n.

36. Suppose that the function $h : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives and satisfies the equation

$$h(x, y) = a\frac{\partial h}{\partial x}(x, y) + b\frac{\partial h}{\partial y}(x, y)$$

for some constants a, b. Prove that if there is a constant M such that $|h(x, y)| \leq M$ for all $(x, y) \in \mathbb{R}^2$, then h is identically zero.

37. Let $f : [0, \infty) \to \mathbb{R}$ be a strictly decreasing continuous function such that $\lim_{x\to\infty} f(x) = 0$. Prove that $\int_0^\infty \frac{f(x)-f(x+1)}{f(x)}\,dx$ diverges.

38. Is there a strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = f(f(x))$ for all x?