PROBLEMS ON ABSTRACT ALGEBRA

♡ 1. Let G be any finite group, and let $w \in G$. Find the number of pairs $(u, v) \in G \times G$ satisfying $w = uvu^2vuv$.

2. (68P) A is a subset of a finite group G (with group operation called multiplication), and A contains more than one half of the elements of G. Prove that each element of G is the product of two (not necessarily distinct) elements of A.

3. (69P) Show that a finite group cannot be the union of two of its proper subgroups. Does the statement remain true if "two" is replaced by "three"?

4. (71P) Let S be a set and let $\circ$ be a binary operation on S satisfying the two laws

$$\begin{aligned} x \circ x &= x \text{ for all } x \text{ in } S, \text{ and} \\ (x \circ y) \circ z &= (y \circ z) \circ x \text{ for all } x, y, z \text{ in } S. \end{aligned}$$

Show that $\circ$ is associative and commutative.

5. (72P) Let S be a set and let $*$ be a binary operation on S satisfying the laws

$$\begin{aligned} x * (x * y) &= y \qquad \text{for all } x, y \text{ in } S, \\ (y * x) * x &= y \qquad \text{for all } x, y \text{ in } S. \end{aligned}$$

Show that $*$ is commutative but not necessarily associative.

6. (72P) Let A and B be two elements in a group such that $ABA = BA^2B$, $A^3 = 1$ and $B^{2n-1} = 1$ for some positive integer n. Prove $B = 1$.

7. (76P) Suppose that G is a group generated by elements A and B, that is, every element of G can be written as a finite "word" $A^{n_1}B^{n_2}A^{n_3}\cdots B^{n_k}$, where $n_1, \ldots, n_k$ are any integers, and $A^0 = B^0 = 1$ as usual. Also, suppoes that $A^4 = B^7 = ABA^{-1}B = 1$, $A^2 \neq 1$, and $B \neq 1$.
   (a) How many elements of G are of the form C^2 with C in G?
   (b) Write each such square as a word in A and B.

♡♡ 8. (77P, B6) Let H be a subgroup with h elements in a group G. Suppose that G has an element a such that for all x in H, $(xa)^3 = 1$, the identity. In G, let P be the subset of all products $x_1ax_2a\cdots x_na$, with n a positive integer and the x_i's in H.
   (a) Show that P is a finite set.
   (b) Show that, in fact, P has no more than $3h^2$ elements.

9. (78P) A "bypass" operation on a set S is a mapping from $S \times S$ to S with the property

$$B(B(w, x), B(y, z)) = B(w, z) \qquad \text{for all} \quad w, x, y, z \quad \text{in} S.$$

   (a) Prove that $B(a, b) = c$ implies $B(c, c) = c$ when B is a bypass.
   (b) Prove that $B(a, b) = c$ implies $B(a, x) = B(c, x)$ for all x in S when B is a bypass.
   (c) Construct a table for a bypass operation B on a finite set S with the following three properties:

(i) $B(x,x) = x$ for all x in S.
(ii) There exist d and e in S with $B(d,e) = d \neq e$.
(iii) There exist f and g in S with $B(f,g) \neq f$.

10. (79P) Let F be a finite field having an odd number m of elements. Let $p(x)$ be an irreducible (i.e., nonfactorable) polynomial over F of the form

$$x^2 + bx + c, \qquad b, c \in F.$$

For how many elements k in F is $p(x) + k$ irreducible over F?

11. (84P) Prove or disprove the following statement: If F is a finite set with two or more elements, then there exists a binary operation $*$ on F such that for all x, y, z in F,

(i) $x * z = y * z$ implies $x = y$ (right cancellation holds), and
(ii) $x * (y * z) \neq (x * y) * z$ (*no* case of associativity holds).

12. (87P) Let F be the field of p^2 elements where p is an odd prime. Suppose S is a set of $(p^2 - 1)/2$ distinct nonzero elements of F with the property that for each $a \neq 0$ in F, exactly one of a and $-a$ is in S. Let N be the number of elements in the intersection $S \cap \{2a : a \in S\}$. Prove that N is even.

♡ 13. (89P) Let S be a nonempty set with an associative operation that is left and right cancellative ($xy = xz$ implies $y = z$, and $yx = zx$ implies $y = z$). Assume that for every a in S the set $\{a^n : n = 1, 2, 3, \dots\}$ is finite. Must S be a group?

♡ 14. (90P) Let G be a finite group of order n generated by a and b. Prove or disprove: there is a sequence

$$g_1, g_2, g_3, \dots, g_{2n}$$

such that

(1) every element of G occurs exactly twice, and
(2) g_{i+1} equals $g_i a$ or $g_i b$ for $i = 1, 2, \dots, 2n$. (Interpret g_{2n+1} as g_1.)

15. (91P) Let P be an odd prime and let $\mathbb{Z}_p$ denote (the field of) integers modulo p. How many elements are in the set

$$\{x^2 : x \in \mathbb{Z}_p\} \cap \{y^2 + 1 : y \in \mathbb{Z}_p\}?$$

16. (92P) Let $\mathcal{M}$ be a set of real $n \times n$ matrices such that

(i) $I \in \mathcal{M}$, where I is the $n \times n$ identity matrix;
(ii) if $A \in \mathcal{M}$ and $B \in \mathcal{M}$, then either $AB \in \mathcal{M}$ or $-AB \in \mathcal{M}$, but not both;
(iii) if $A \in \mathcal{M}$ and $B \in \mathcal{M}$, then either $AB = BA$ or $AB = -BA$;
(iv) if $A \in \mathcal{M}$ and $A \notin I$, there is at least one $B \in \mathcal{M}$ such that $AB = -BA$.

Prove that $\mathcal{M}$ contains at most n^2 matrices.

17. (94P, B6) For any integer a, set

$$n_a = 101a - 100 \cdot 2^a.$$

Show that for $0 \leq a, b, c, d \leq 99$, $n_a + n_b \equiv n_c + n_d \pmod{10100}$ implies $\{a, b\} = \{c, d\}$.

18. (96P, A4) Let S be a set of ordered triples (a, b, c) of distinct elements of a finite set A. Suppose that

   (1) $(a, b, c) \in S$ if and only if $(b, c, a) \in S$;

   (2) $(a, b, c) \in S$ if and only if $(c, b, a) \notin S$ [for a, b, c distinct];

   (3) (a, b, c) and (c, d, a) are both in S if and only if (b, c, d) and (d, a, b) are both in S.

   Prove that there exists a one-to-one function g from A to $\mathbb{R}$ such that $g(a) < g(b) < g(c)$ implies $(a, b, c) \in S$.

19. (97P, A4) Let G be a group with identity e and $\phi : G \to G$ a function such that

$$\phi(g_1)\phi(g_2)\phi(g_3) = \phi(h_1)\phi(h_2)\phi(h_3)$$

   whenever $g_1g_2g_3 = e = h_1h_2h_3$. Prove that there exists an element $a \in G$ such that $\psi(x) = a\phi(x)$ is a homomorphism (that is, $\psi(xy) = \psi(x)\psi(y)$ for all $x, y \in G$).

20. (01P, A1) Consider a set S and a binary operation $*$, i.e., for each $a, b \in S$, $a * b \in S$. Assume $(a * b) * a = b$ for all $a, b \in S$. Prove that $a * (b * a) = b$ for all $a, b \in S$.

21. (07P, A5) Suppose that a finite group has exactly n elements of order p, where p is a prime. Prove that either $n = 0$ or p divides $n + 1$.

22. (08P, A6) Prove that there exists a constant $c > 0$ such that in every nontrivial finite group G there exists a sequence of length at most $c \ln |G|$ with the property that each element of G equals the product of some subsequence. (The elements of G in the sequence are not required to be distinct. A *subsequence* of a sequence is obtained by selecting some of the terms, not necessarily consecutive, without reordering them; for example, $4, 4, 2$ is a subsequence of $2, 4, 6, 4, 2$, but $2, 2, 4$ is not.)

23. (09P, A5) Is there a finite abelian group G such that the product of the orders of all its elements is 2^{2009}?

24. (10P, A5)

   Let G be a group, with operation $*$. Suppose that

   (a) G is a subset of $\mathbb{R}^3$ (but $*$ need not be related to addition of vectors);

   (b) For each $\mathbf{a}, \mathbf{b} \in G$, either $\mathbf{a} \times \mathbf{b} = \mathbf{a} * \mathbf{b}$ or $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ (or both), where $\times$ is the usual cross product in $\mathbb{R}^3$.

   Prove that $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ for all $\mathbf{a}, \mathbf{b} \in G$.

♡♡ 25. (11P, A6) Let G be an abelian group with n elements, and let $\{g_1 = e, g_2, \ldots, g_k\} \subsetneq G$ be a (not necessarily minimal) set of distinct generators of G. A special die, which randomly selects one of the elements $g_1, g_2, \ldots, g_k$ with equal probability, is rolled m times and the selected elements are multiplied to produce an element $g \in G$.

   Prove that there exists a real number $b \in (0, 1)$ such that

$$\lim_{m\to\infty} \frac{1}{b^{2m}} \sum_{x \in G} \left(\mathrm{Prob}(g = x) - \frac{1}{n} \right)^2$$

   is positive and finite.

26. (12P, A2) Let $*$ be a commutative and associative binary operation on a set S. Assume that for every x and y in S, there exists z in S such that $x * z = y$. (This z may depend on x and y.) Show that if a, b, c are in S and $a * c = b * c$, then $a = b$.

27. (16P, A5) Suppose that G is a finite group generated by the two elements g and h, where the order of g is odd. Show that every element of G can be written in the form

$$g^{m_1} h^{n_1} g^{m_2} h^{n_2} \cdots g^{m_r} h^{n_r}$$

with $1 \leq r \leq |G|$ and $m_1, n_1, m_2, n_2, \ldots, m_r, n_r \in \{1, -1\}$.

♡♡ 28. (18P, A4) Let m and n be positive integers with $\gcd(m, n) = 1$, and let

$$a_k = \lfloor mk/n \rfloor - \lfloor m(k-1)/n \rfloor$$

for $k = 1, 2, \ldots, n$. Suppose that g and h are elements in a group G and that

$$gh^{a_1} gh^{a_2} \ldots gh^{a_n} = e,$$

where e is the identity element. Show that $gh = hg$.

♡♡ 29. Let x, y be elements in a (not necessarily commutative) ring such that $1 - xy$ is invertible. Prove that $1 - yx$ is also invertible.

HINT:

30. Let R be a *noncommutative* ring with identity. Suppose that x, y are elements of R such that $1 - xy$ and $1 - yx$ are invertible. (By the previous problem it suffice to assume that only $1 - xy$ is invertible, but this is irrelevant.) Show that

$$(1 + x)(1 - yx)^{-1}(1 + y) = (1 + y)(1 - xy)^{-1}(1 + x). \tag{1}$$

This problem illustrates that "noncommutative high school algebra" is a lot harder than ordinary (commutative) high school algebra.

NOTE. Formally we have

$$(1 - yx)^{-1} = 1 + yx + yxyx + yxyxyx + \cdots$$

and similarly for $(1 - xy)^{-1}$. Thus both sides of (1) are formally equal to the sum of all "alternating words" (products of x's and y's with no two x's or y's appearing consecutively). This makes the identity (1) plausible, but our formal argument is not a proof.

31. Let G be a finite abelian group of order n. Suppose that for each prime divisor p of n, there is exactly one subgroup of G of order p. Prove that G is a cyclic group.

32. Prove that there is no nontrivial automorphism of the ring of real numbers. That is, if $f : \mathbb{R} \to \mathbb{R}$ is a function such that $f(0) = 0$, $f(1) = 1$, $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$, and $f(xy) = f(x)f(y)$ for all $x, y \in \mathbb{R}$, then $f(x) = x$ for all $x \in \mathbb{R}$.

33. (a) Let G be a finitely generated group in which $g^2 = 1$ for each $g \in G$. Prove that G is finite and abelian.

(b) Let G be a finitely generated group in which $g^3 = 1$ for each $g \in G$. Prove that G is finite.

(Beware that (b) is hard, while its analogue with 3 replaced by an arbitrary positive integer is in fact false!)

34. Let G be a group of order $4n + 2$, $n \geq 1$. Prove that G is not a simple group, i.e., G has a proper normal subgroup.

35. Let R be a noncommutative ring with identity. Show that if an element $x \in R$ has more than one right inverse (i.e., there is more than one $y \in R$ such that $xy = 1$), then x has infinitely many right inverses.

♡ 36. Let R be a ring for which $x^2 = 0$ for all $x \in R$. Show that $xyz + xyz = 0$ for all $x, y, z \in R$.

37. Let R satisfy all the axioms of a ring except commutativity of addition. Show that $ax + by = by + ax$ for all $a, b, x, y \in R$.

38. How many $n \times n$ matrices of rank r are there over the finite field $\mathbb{F}_q$?

♡♡ 39. Let G denote the set of all infinite sequences $(a_1, a_2, \dots)$ of integers a_i. We can add elements of G coordinate-wise, i.e.,

$$(a_1, a_2, \dots) + (b_1, b_2, \dots) = (a_1 + b_1, a_2 + b_2, \dots).$$

Let $\mathbb{Z}$ denote the set of integers. Suppose $f : G \to \mathbb{Z}$ is a function satisfying $f(x + y) = f(x) + f(y)$ for all $x, y \in G$.

(a) Let e_i be the element of G with a 1 in position i and 0's elsewhere. Suppose that $f(e_i) = 0$ for all i. Show that $f(x) = 0$ for all $x \in G$. (Note. From the fact that f preserves the sum of two elements it follows easily that f preserves *finite* sums. However, it does not necessarily follow that f preserves *infinite* sums.)

(b) Show that $f(e_i) = 0$ for all but finitely many i.

40. Let G be a finite group, and set $f(G) = \#\{(u, v) \in G \times G : uv = vu\}$. Find a formula for $f(G)$ in terms of the order of G and the number $k(G)$ of conjugacy classes of G. (Two elements $x, y \in G$ are *conjugate* if $y = axa^{-1}$ for some $a \in G$. Conjugacy is an equivalence relation whose equivalence classes are called *conjugacy classes*.)

41. (difficult) Let n be an odd positive integer. Show that the number of ways to write the identity permutation ι of $1, 2, \dots, n$ as a product $uvw = \iota$ of three n-cycles is $2(n-1)!^2/(n+1)$.

42. Show that the number of ways to write the cycle $(1, 2, \dots, n)$ as a product of $n - 1$ transpositions is n^{n-2}. For instance, when $n = 3$ we have (multiplying permutations left-to-right) three ways:

$$(1, 2, 3) = (1, 3)(2, 3) = (1, 2)(1, 3) = (2, 3)(1, 2).$$

43. (difficult) Let $s_i = (i, i + 1) \in S_n$, i.e., s_i is the permutation of $1, 2, \dots, n$ that transposes i and $i + 1$ and fixes all other j. Let $f(n)$ be the number of ways to write the permutation $n, n - 1, \dots, 1$ in the form $s_{i_1} s_{i_2} \cdots s_{i_p}$, where $p = \binom{n}{2}$. For instance, $321 = s_1 s_2 s_1 = s_2 s_1 s_2$, so $f(3) = 2$. Moreover, $f(4) = 16$. Show that $f(n)$ is the number of sequences $a_1, \dots, a_p$ of $n-1$ 1's, $n-2$ 2's, $\dots$, one $n-1$, such that in any prefix $a_1, a_2, \dots, a_k$, the number of $i+1$'s

does not exceed the number of i's. For instance, when $n = 3$ there are the two sequences 112 and 121.

NOTE. An explicit formula is known for $f(n)$, but this is irrelevant here.

44. (difficult) In the notation of the previous problem, show that

$$\sum_{i_1,i_2,\ldots,i_p} i_1 i_2 \cdots i_p = p!,$$

where the sum is over all sequences $i_1, \ldots, i_p$ for which $n, n-1, \ldots, 1 = s_{i_1} s_{i_2} \cdots s_{i_p}$. For instance, when $n = 3$ we get $1 \cdot 2 \cdot 1 + 2 \cdot 1 \cdot 2 = 3!$.

NOTE. The only known proofs are algebraic. It would be interesting to give a combinatorial proof.