Problem Set 2. Due 9/18

1. Prove that for all positive integers n, the number $\binom{2n}{n}$ is even.

2. Prove that for all positive integers n,
$$n\binom{2n-1}{n-1} = \sum_{k=1}^{n} k\binom{n}{k}^2.$$

3. Find a closed formula for
$$\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}t^{k+1},$$
where n is a positive integer and t is a real number.

4. What digits are immediately on the left and right of the decimal point in $(\sqrt{11}+\sqrt{10})^{2018}$? *(You should not use any computer/calculator for this problem. Hint: also consider $(\sqrt{11}-\sqrt{10})^{2018}$)*

5. Prove that the number of partitions of the integer n is equal to the number of partitions of the integer $2n$ with no odd parts.

6. Let a_n be the number of compositions of n into parts that are larger than 1. Determine a formula for a_n in terms of the Fibonacci numbers. (The Fibonacci numbers F_n are defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 3$.)

7. Let a_n denote the number of partitions of $[n]$ so that no two consecutive integers belong to the same block. Prove that $a_n = B(n-1)$, the $(n-1)$-th Bell number.

8. Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a partition of n and $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ be its conjugate. Prove that
$$\sum_{i\geq 1}\left\lfloor \frac{\lambda_{2i-1}}{2}\right\rfloor = \sum_{i\geq 1}\left\lceil \frac{\lambda'_{2i}}{2}\right\rceil.$$
Here, for $x \in \mathbb{R}$, $\lfloor x \rfloor$ is the greatest integer less than or equal x, and $\lceil x \rceil$ is the least integer greater than or equal to x.