Problem Set 10. Due 12/4

Reminder: You must acknowledge your sources and collaborators (even if it is "none", you must write so). Failure to do so on this problem set will result in an automatic 2-point deduction.

1. Let G be a graph on n vertices and $3n - 6 + k$ edges for some $k > 0$. Prove that any drawing of G in the plane contains at least k crossing pairs of edges.

2. Let G be a 2-connected graph on $n \geq 5$ vertices that does not contain a $K_{2,3}$-subdivision.
(a) Show that G does not contain a K_4-subdivision.
(b) Deduce that G has at most $2n - 3$ edges.

3. Prove or disprove: every graph G has a proper $\chi(G)$-coloring where $\alpha(G)$ vertices receive the same color. ($\alpha(G)$ is the size of the largest independent set of G)

4. Prove that $\chi(G) = \omega(G)$ when the complement of G is bipartite.

5. Let G be a graph in which every pair of odd cycles has a common vertex. Show that $\chi(G) \leq 5$.

6. Let G be a union of k forests. Prove that $\chi(G) \leq 2k$.

7. Let G be a graph in which every edge belongs to at most k cycles. Show that $\chi(G) \leq k + 2$.

Hints:

2. (b) Add a new vertex and join it to all vertices of G and apply Kuratowski's theorem

7. Consider a longest path and greedily color one of its endpoints.