Practice Midterm 3

Time: 80 minutes.

5 problems worth 10 points each.

No electronic devices. You may bring **two sheets of notes** on letter-sized paper (total four sides front and back) **in your own handwriting**. Typed, printed, or photocopied notes are **forbidden**.

You must provide justification in your solutions (not just answers). You may quote theorems and facts proved in class, course textbook/notes, or homework, provided that you state the facts that you are using.

1. Determine whether each of the following statement is TRUE or FALSE, and provide a short justification or a counterexample (a correct answer without justification receives zero credit).

   (a) If G is a connected planar graph, then any planar embedding of G always has the same number of faces.

   Solution. True By Euler's formula, $v - e + f = 2$, and v and e are determined by the graph.

   (b) If G is a connected d-regular graph with $d \geq 1$, then its line graph $L(G)$ contains an Eulerian tour.

   Solution. True $L(G)$ is connected since G is connected, and every vertex of $L(G)$ has degree $2(d-1)$, which is even.

2. Does there exist a connected graph with a cut vertex whose edge set can be partitioned into perfect matchings?

   Solution. No. Let G be a graph with cut vertex v, and let $C_1, \dots, C_k$ ($k \geq 2$) be components of $G - v$. Let u_i be a neighbor of v in C_i.

   If C_1 is odd, then vu_2 cannot be contained in a perfect matching since it would give rise to a perfect matching in C_1, which has an odd number of vertices. Likewise, if C_1 is even, then vu_1 cannot be contained in a perfect matching since it would give arise to a perfect matching in $C_1 - u_1$, which has an odd number of vertices. Therefore, not every edge of G can be contained in a perfect matching, and in particular G cannot be partitioned into perfect matchings.

3. Let G be a bipartite graph with n vertices on both sides and minimum degree at least $n/2$. Prove that G has a perfect matching.

   Solution. Let $A \cup B$ be a vertex bipartition of G. We would like to check the condition in Hall's theorem. Let $S \subset A$ be nonempty. Since every vertex has degree at least $n/2$, $|N(S)| \geq n/2$. So $|N(S)| \geq |S|$ whenever $|S| \leq n/2$. So assume that $|S| > n/2$. Then every vertex in B, having degree at least $n/2$, is adjacent to some vertex of S. Hence $|N(S)| = |B| = n \geq |S|$. Thus by Hall's theorem G has a perfect matching.

4. Let $k \geq 1$. Let G be a $2k$-edge-connected graph. Let $s_1, \dots, s_k, t_1, \dots, t_k$ be distinct vertices. Show that there are edge disjoint paths $P_1, \dots, P_k$ such that each P_i starts at s_i and ends at t_i.

   Solution. Since G is $2k$-edge-connected, every vertex has degree at least $2k$ (otherwise we can disconnect a vertex by removing fewer than $2k$ edges), so there is another vertex v different from $s_1, \dots, s_k, t_1, \dots, t_k$.

   Let G' be the graph obtained from G by adding a new vertex v and making it adjacent to all s_i's and t_i's. Then G' is also $2k$-edge-connected. By the edge-version of Menger's theorem (Corollary 3.25 in the notes), there exist $2k$ edge-disjoint paths between u and v in G'. In other words, there exist a collection of edge-disjoint paths $Q_1, \dots, Q_k, Q'_1, \dots, Q'_k$ in G where Q_i is v-s_i path and Q'_i is a v-t_i path. Let $P_i = Q_i \cup Q'_i$. Then the P_i's are the desired paths from s_i to t_i.

5. Prove that the union of k planar graphs is $6k$-colorable.

   Solution. Recall that every planar graph on n vertices has at most $3n - 6$ edges (a corollary of Euler's formula), and hence average degree strictly less than 6. Hence a union of k planar graphs has average degree less than $6k$, and hence minimum degree less than $6k$.

   Every subgraph of a union of k planar graphs is still a union of k planar graphs. Hence every union of k planar graphs is $(6k - 1)$-degenerate, and thus $6k$-colorable (by greedy coloring, c.f., Theorem 7.19 in the notes).