Practice Midterm 1

Closed book. No notes/calculators/phones.
Time: 80 minutes.
6 problems worth 10 points each.
You must provide justification in your solutions (not just answers). Simplify all answers and express in closed form whenever possible.

1. Let $n \geq 3$ be a positive integer. Determine the number of solutions to $x + y + z \leq n$ with integers $x, y, z \geq 1$.

   Solution. Let $a = x - 1$, $b = y - 1$, $c = z - 1$, and introduce an additional "slack" variable d. There is a bijection with solutions to $a + b + c + d = n - 3$ with nonnegative integers a, b, c, d. This is the number of weak compositions of $n - 3$ into four parts, which we solved in class using a "stars and bars" argument (counting linear arrangements of $n - 3$ stars and 3 bars). Thus number of solutions is $\boxed{\binom{n}{3}}$.

2. Prove that for all positive integers n,

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

   Solution. Consider the number of ways to choose a committee of n persons from n men and n women. There are $\binom{2n}{n}$ ways.

   On the other hand, for each $k = 0, \ldots, n$, the number of ways of forming the committee using k men and $n - k$ women is exactly $\binom{n}{k}\binom{n}{n-k} = \binom{n}{k}^2$, and summing over k gives the total number of ways.

3. Let $D(n)$ denote the number of derangements (permutations without fixed points) of $[n]$. Give a combinatorial proof of the identity

$$D(n+1) = n(D(n) + D(n-1)), \quad \text{for all } n \geq 1.$$

   Do not use the formula for the numbers $D(n)$ derived in class.

   Solution. Let us count the number of derangements of $[n+1]$ by considering the cycle that the number $n + 1$ lies in.

   Case 1: $n + 1$ lies in a cycle of length 2. There are n choices for the other number in the same cycle as $n+1$. After removing these two elements, the number of ways to permuting the remaining elements without fixed points is $D(n-1)$. Thus there are $nD(n-1)$ derangements where $n + 1$ lies in a 2-cycle.

   Case 2: $n + 1$ lies in a cycle of length greater than 2. Consider the cycle decomposition. If we remove the number $n + 1$ from its cycle, it does not result in a fixed point (since the cycle containing the number $n + 1$ has length at least 3), so we are left with a derangement of $[n]$. On the other hand, for each derangement of $[n]$, viewed as a cycle decomposition, there are n places where we can insert the number $n + 1$ into one of the existing cycles (by picking that number the comes before the spot where we would like to insert $n+1$ in the cycle decomposition). Thus there are $nD(n)$ derangements where the number $n + 1$ lies in a cycle of length greater than 2.

4. Let $n \geq 4$ be a positive integer. How many permutations of $[n]$ are there such that some cycle contains both 1 and 2 and a different cycle contains both 3 and 4?

 Solution. Let us relabel elements of $[n]$ so that $1, 2, 3, 4$ become $n-3, n-2, n-1, n$ respectively. Consider the canonical cycle form and dropping the parentheses. We have n and $n-1$ in one cycle and $n-2$ and $n-3$ in a different cycle if and only if the numbers $n-2, n-3, n, n-1$ must appear in this order. Indeed, since the cycles are listed with its largest element first, and in increasing order of its initial element, the rightmost cycle starts with n and must contain $n-1$, and the second-rightmost cycle must start with $n-2$ and must contain $n-1$.

 We proved in class that the operation of writing a permutation in its canonical cycle form and dropping the parentheses is a bijection on the set of permutations of $[n]$. The numbers $n-3, n-2, n-1, n$ are equally likely to appear in each of the $4! = 24$ orders among all $n!$ permutations. Therefore, the number of permutation have $n-2, n-3, n, n-1$ appearing in this specific order is $\boxed{n!/24}$, which is also the answer to the original question due to the bijection.

5. Let $a_0 = 0$ and $a_{n+1} = 3a_n + n$ for all $n \geq 0$.

 (a) Express the generating function $A(x) = \sum_{n \geq 0} a_n x^n$ in closed form.

 (b) Find a closed form formula for a_n.

 Solution. (a) Multiplying the recurrence by x^{n+1} and summing over all $n \geq 0$, we have

$$\sum_{n \geq 0} a_{n+1} x^{n+1} = \sum_{n \geq 0} 3a_n x^{n+1} + \sum_{n \geq 0} n x^{n+1}.$$

 We have

$$\sum_{n \geq 0} a_{n+1} x^{n+1} = A(x) - a_0 = A(x)$$

$$\sum_{n \geq 0} 3a_n x^{n+1} = 3xA(x)$$

 and

$$\sum_{n \geq 0} n x^{n+1} = x^2 \frac{d}{dx} \sum_{n \geq 0} x^n = x^2 \frac{d}{dx} \left(\frac{1}{1-x} \right) = \frac{x^2}{(1-x)^2}.$$

 Thus

$$A(x) = 3xA(x) + \frac{x^2}{(1-x)^2}.$$

 Solving for $A(x)$, we find

$$\boxed{A(x) = \frac{x^2}{(1-3x)(1-x)^2}}.$$

 (b) We have the following partial fraction decomposition:

$$\begin{aligned} A(x) &= \frac{1/4}{1-3x} + \frac{1/4}{1-x} - \frac{1/2}{(1-x)^2} \\ &= \sum_{n \geq 0} \left(\frac{3^n}{4} + \frac{1}{4} - \frac{n+1}{2} \right) x^n. \\ &= \sum_{n \geq 0} \left(\frac{3^n - 2n - 1}{4} \right) x^n. \end{aligned}$$

Thus

$$\boxed{a_n = \frac{3^n - 2n - 1}{4}, \quad \text{for all } n \geq 0.}$$

6. Let a_n be the number of partitions of n whose parts differ by at least two. For instance, when $n = 10$ the partitions are (10), $(9,1)$, $(8,2)$, $(7,3)$, $(6,4)$, $(6,3,1)$.

   Let b_n be the number of partitions of n whose smallest part is at least as large as the number of parts. For instance, when $n = 10$ the partitions are (10), $(8,2)$, $(7,3)$, $(6,4)$, $(5,5)$, $(4,3,3)$.

   Give a bijective proof that $a_n = b_n$.
   Hint. Consider $1 + 3 + 5 + \cdots + (2k-1)$.

   Solution. Consider a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n whose smallest part λ_k satisfies $\lambda_k \geq k$. Let $\mu = (\lambda_1 - k, \lambda_2 - k, \ldots, \lambda_k - k)$. Note that $1 + 3 + 5 + \cdots + (2k-1) = k^2$. Let

$$\nu = (\mu_1 + 2k - 1, \mu_2 + 2k - 3, \mu_3 + 2k - 5, \ldots, \mu_{k-1} + 3, \mu_k + 1).$$

   Then ν is a partition of n whose parts differ by at least two.

   The process can be reversed. Starting with partition $\nu = (\nu_1, \ldots, \nu_k)$ satisfying $\nu_i - \mu_{i+1} \geq 2$ for each $1 \leq i \leq k-1$. We have $\nu_k \geq 1$, $\nu_{k-1} \geq 3$, $\ldots$, $\nu_1 \geq (2k-1)$. Let

$$\mu = (\nu_1 - (2k-1), \nu_2 - (2k-3), \ldots, \nu_k - 1).$$

   (This is the same μ as in the forward map!) And let $\lambda = (\mu_1 + k, \mu_2 + k, \ldots, \mu_k + k)$. Then λ is a partition of n with $\lambda_k \geq k$, and this map is the inverse of the earlier map.

   This gives the desired bijection.