Quasirandom Cayley Graphs

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Joint work with David Conlon

Quasirandom graphs

Theorem (Chung—Graham—Wilson 1989)**.**
Let G be a d-regular graph on n vertices with $d = \Theta(n)$.
The following two properties are equivalent:

- **Discrepancy:** For all subsets S and T of vertices in G,
$$e(S,T) = \frac{d}{n}|S||T| + o(nd)$$
- **Eigenvalue:** All eigenvalues except for the largest are $o(d)$.

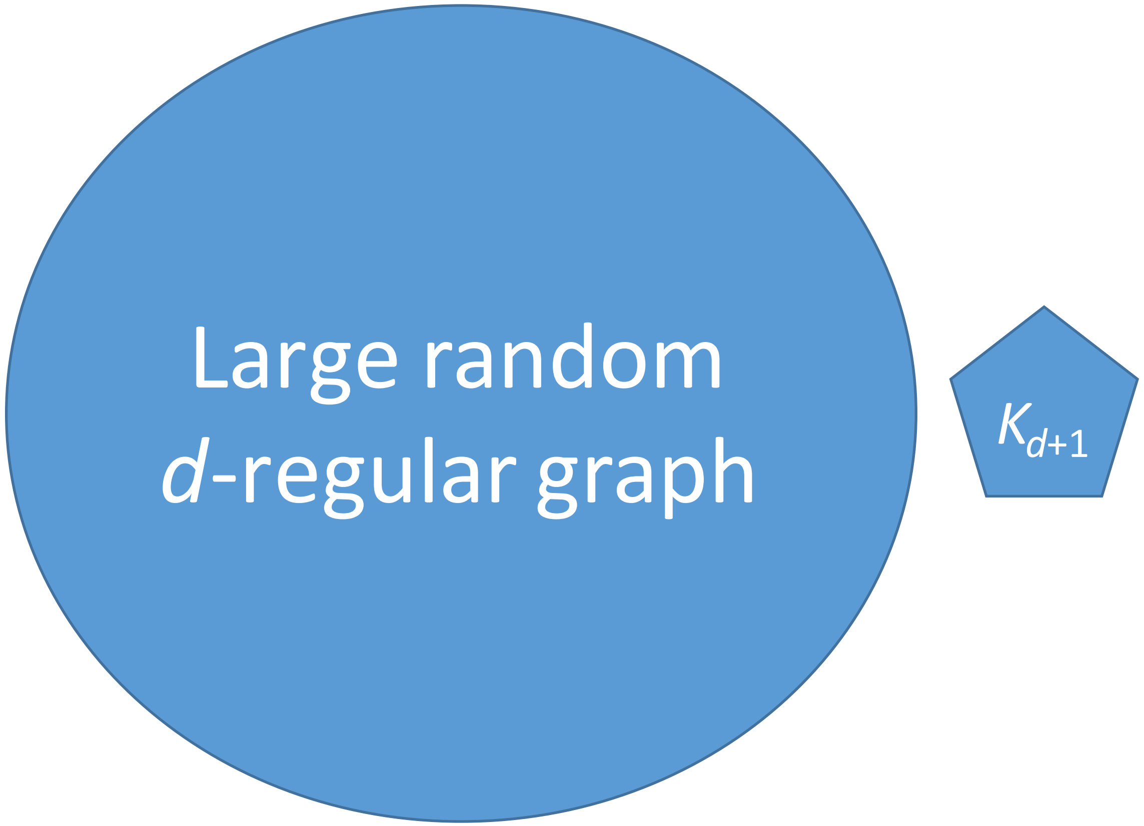
What about when $d = o(n)$?

Eigenvalue implies discrepancy

- (n,d,λ)-graph: n-vertex, d-regular, all eigenvalues except the largest are bounded in absolute value by λ
- **Expander mixing lemma:**
  In any (n,d,λ)-graph, for any vertex subsets S and T,
$$\left|e(S,T)-\frac{d}{n}|S||T|\right| \le \lambda\sqrt{|S||T|} \le \lambda n$$
- Thus small second eigenvalue (i.e., $\lambda = o(d)$) implies small discrepancy
- How about the converse?

Graphs with low discrepancy and high second eigenvalue

[Krivelevich—Sudakov 2006, Bollobás—Nikiforov 2004]



Cayley graphs

- Let G be a finite group, and S a subset of elements.
- Cay(G,S) is the graph with vertex set G and edges (g, gs)
- Many well-known expander graphs are Cayley graphs, e.g., Ramanujan graphs of Lubotzky—Philips—Sarnak / Margulis

Quasirandom Cayley graphs

Theorem (Kohayakawa—Rödl—Schacht).
For Cayley graphs of **abelian** groups, discrepancy condition is equivalent to eigenvalue condition.

Theorem (Conlon—Z.).
The same is true for all Cayley graphs.

- As a corollary, the same is true for all vertex-transitive graphs.
- In fact, the equivalence holds with linear dependence of parameters

Quasirandom Cayley graphs

Definitions: (n,d,λ)-graph: n-vertex, d-regular, $|\lambda_2|, |\lambda_n| \le \lambda$.

An n-vertex d-regular graph is ϵ-uniform if

$$\left| e(S,T) - \frac{d}{n}|S||T| \right| \le \epsilon dn$$

for all vertex subsets S and T (quasirandom corresponds to $\epsilon = o(1)$).

Main theorem (Conlon—Z.).
Every ϵ-uniform Cayley graph is an (n,d,λ)-graph with $\lambda \le 14\epsilon d$.

Corollary.
Same is true for all vertex-transitive graphs. (Also bipartite analogs)

Equivalent norms on groups

Let $f: G \to \mathbb{C}$, define the cut norm:

$$\|f\|_{\mathrm{cut}} = \sup_{S,T \subseteq G} \left|\mathbb{E}_{g,h\in G}\, f(gh^{-1}) 1_S(g) 1_T(h)\right|$$

and the spectral norm:

$$\|f\| = \sup_{\substack{x,y: G\to\mathbb{C} \\ \|x\|_2, \|y\|_2 \le 1}} \left|\mathbb{E}_{g,h\in G}\, f(gh^{-1}) \overline{x(g)} y(h)\right|$$

Claim. $\|f\|_{\mathrm{cut}} \le \|f\| \le 14\|f\|_{\mathrm{cut}}$

Norms on abelian groups

$$\|f\|_{\infty\to 1} = \sup_{x,y:G\to\mathbb{D}} \left|\mathbb{E}_{g,h\in G}\, f(gh^{-1})\overline{x(g)}y(h)\right|$$

$$\|f\|_{\text{cut}} \le \|f\|_{\infty\to 1} \le \pi^2\|f\|_{\text{cut}}$$

[Kohayakawa—Rödl—Schacht, proof attributed to Gowers]

Theorem. $\|f\|_{\infty\to 1} = \|f\|$ for any function f on an abelian group.

Proof. Easy direction: $\|f\|_{\infty\to 1} \le \|f\|$. Other direction:

$$\|f\| = \sup_{\chi\in\hat{G}} |\langle f,\chi\rangle|$$

Set $x = y = \chi$ in definition of $\|f\|_{\infty\to 1}$.

Remark. In general, for non-abelian G, $\|f\|_{\infty\to 1} \ne \|f\|$

Semidefinition relaxation

Grothendieck norm of a function $f: G \to \mathbb{C}$:

$$\|f\|_G = \sup_{x,y:G\to B(\mathbb{H})} \left|\mathbb{E}_{g,h\in G} f(gh^{-1})\langle x(g), y(h)\rangle\right|$$

$\mathbb{H}$ is an arbitrary complex Hilbert space. Wlog, $\mathbb{H} = \mathbb{C}^{2n}$

Grothendieck's inequality:

$$\|f\|_G \leq K\|f\|_{\infty\to 1}$$

for some constant $K \leq 1.401$ [Haagerup 1987].

See [Braverman—Makarychev—Makarychev—Naor 2013] for the best bound in the real setting

Theorem (Conlon—Z.). $\|f\|_G = \|f\|$ for any function $f: G \to \mathbb{C}$.

Related works

- [Alon—Coja-Oghlan—Han—Kang—Rödl—Schacht 2007]
  If any graph has small discrepancy, then one can remove an ϵ-fraction of vertices to eliminate all large eigenvalues (except for the largest)

  Proof uses Grothendieck's inequality and SDP duality

- [Gowers 2008] Quasirandom groups
  If a group has no small nontrivial representations, then all of its Cayley graphs are quasirandom

Non-abelian Fourier transform:

For each irreducible representation $\rho \in \hat{G}$, we have a $d_\rho \times d_\rho$ matrix

$$\hat{f}(\rho) = \mathbb{E}_{g \in G} f(g) \rho(g)$$

Inversion formula: $f(g) = \sum_{\chi \in \hat{G}} d_\rho \mathrm{Tr}\left(\hat{f}(\rho)\rho(g)^*\right)$

Spectral norm: $\|f\| = \max_{\rho \in \hat{G}} \left\|\hat{f}(\rho)\right\|$

Proof sketch of $\|f\|_G = \|f\|$.

- Apply SVD to the Fourier transform $\hat{f}(\rho)$.
- Let $x(g)$ and $y(g)$ be the images of the top singular vectors under the action of the representation ρ
- Apply Schur's lemma for orthogonality

$$\|f\|_G = \sup_{x,y:G \to B(\mathbb{H})} \left| \mathbb{E}_{g,h \in G} f(gh^{-1}) \langle x(g), y(h) \rangle \right|$$