Extremal results for sparse pseudorandom graphs

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Joint work with David Conlon and Jacob Fox

Sparse extensions

Extending classical results to sparse settings. For example:

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Szemerédi's Theorem

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The primes contain arbitrarily long arithmetic progressions

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Every subset of the integers with positive density contains arbitrarily long arithmetic progressions

Green-Tao Theorem

The primes contain arbitrarily long arithmetic progressions

- The primes have zero density, but there is a pseudorandom set of "almost primes" in which the primes form a subset with positive relative density.
- Transference principle: dense $\to$ sparse.

Sparse setting

Dense setting

Host graph: K_n

G: arbitrary dense graph

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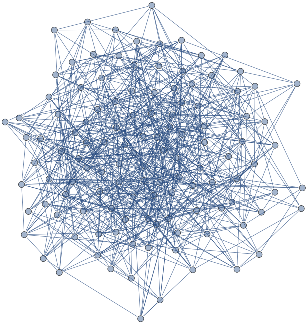
Sparse setting

Host graph: pseudorandom graph with $\Omega(n^{2-c})$ edges



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Dense setting

Host graph: K_n

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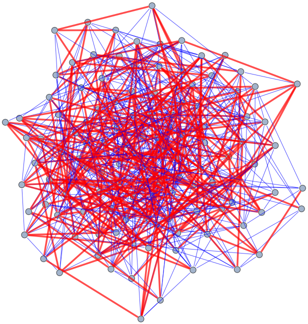
Sparse setting

Host graph: pseudorandom graph with $\Omega(n^{2-c})$ edges

G: relatively dense subgraph



Szemerédi's Regularity Lemma

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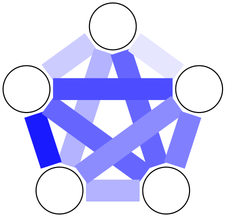
Roughly speaking, every large graph can be partitioned into a bounded number of roughly equally-sized parts so that the graph is random-like between most pairs of parts.



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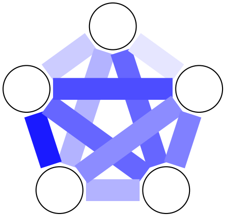

Regularity Method

1. Apply Szemerédi's Regularity Lemma.
2. Apply a counting lemma for embedding small graphs.

Regular partition

Edge density: $d_G(U, V) = \frac{e_G(U,V)}{|U||V|}$.

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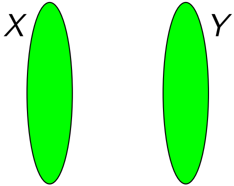
Definition (ϵ-regular)

Bipartite graph $(X, Y)_G$ is ϵ-regular if for all $A \subset X$, $B \subset Y$, with $|A| \geq \epsilon |X|$ and $|B| \geq \epsilon |Y|$, we have

$$|d_G(A, B) - d_G(X, Y)| < \epsilon \, .$$



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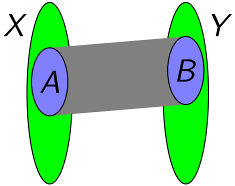

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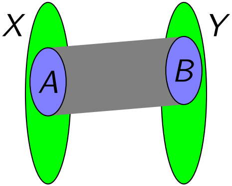

Definition (ϵ-regular partition)

A partition of vertices into nearly-equal parts where all but ϵ-fraction of the pairs of parts induce ϵ-regular bipartite graphs.

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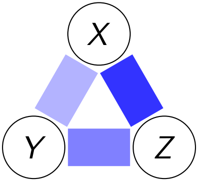
Triangle counting lemma

If G is a tripartite graph that is ϵ-regular between each pair of parts, then the number of triangles in G is $\approx d_G(X,Y)d_G(Y,Z)d_G(X,Z)\,|X|\,|Y|\,|Z|$.



Sparse regularity

- Original regularity method applies only for dense graphs.
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Open problem

A counting lemma for sparse regular graphs.

- Previous work:
  counting triangles [Kohayakawa, Rödl, Schacht & Skokan '10]

Main result

Sparse Counting Lemma [Conlon–Fox–Z.]

For any graph H, there is a counting lemma for embedding H into a regular partition in a sparse pseudorandom graph.

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Applications

Sparse extensions of:

- Turán, Erdős-Stone-Simonovits
- Ramsey
- Graph removal lemma

  . . .

Pseudorandom graphs

Definition

We say that a graph Γ is (p, β)-jumbled if for all vertex subsets X and Y of Γ, we have

$$\left| e(X, Y) - p\,|X|\,|Y| \right| \leq \beta \sqrt{|X|\,|Y|}.$$

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Examples

- Random graph $G(n, p)$ is (p, β)-jumbled with $\beta = O(\sqrt{np})$ w.h.p.
- (n, d, λ)-graph is $(\frac{d}{n}, \lambda)$-jumbled by expander mixing lemma.

Turán-type results

Turán's Theorem

Any K_r-free graph on n vertices has at most $(1 - \frac{1}{r-1})\frac{n^2}{2}$ edges.

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Erdős-Stone-Simonovits Theorem

For any fixed H, any H-free graph on n vertices has at most

$$\left(1 - \frac{1}{\chi(H) - 1} + o(1)\right)\binom{n}{2}$$

edges, where $\chi(H)$ is the chromatic number of H.

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Sparse extension: replace K_n by a jumbled graph Γ.

Sparse extensions of Erdős-Stone-Simonovits

Previous work:

- $H = K_t$ [Sudakov, Szabó & Vu '05] [Chung '05]
- H triangle-free [Kohayakawa, Rödl, Schacht, Sissokho, Skokan '07]

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Sparse Erdős-Stone-Simonovits [Conlon–Fox–Z.]

For every graph H and every $\epsilon > 0$, there exists $c > 0$ such that if $\beta \le c p^{d(H)+\frac{5}{2}} n$ then any (p, β)-jumbled graph Γ on n vertices has the property that any H-free subgraph of Γ has at most

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The difficulty with pseudorandom graphs

- Alon (1994) constructed a triangle-free (n, d, λ)-graph with $\lambda \le c\sqrt{d}$ and $d \ge n^{2/3}$.
- I.e., there exists a $(p, cp^2 n)$-jumbled graph Γ containing no triangles.

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- I.e., there exists a $(p, cp^2 n)$-jumbled graph Γ containing no triangles.
- $\rightarrow$ No counting lemma for Γ
- $\rightarrow$ Extensions of applications *false* for Γ

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For any graph H and positive integer r, if n is sufficiently large, then any r-coloring of the edges of K_n contains a monochromatic copy of H.

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Sparse Ramsey [Conlon–Fox–Z.]

For every graph H and every positive integer $r \geq 2$, there exists $c > 0$ such that if $\beta \leq cp^{d(H)+\frac{5}{2}}n$ then any (p, β)-jumbled graph Γ on n vertices has the property that any r-coloring of the edges of Γ contains a monochromatic copy of H.

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Removal lemmas

Triangle Removal Lemma [Ruzsa & Szemerédi '78]

Every graph on n vertices with at most $o(n^3)$ triangles can be made triangle-free by deleting at most $o(n^2)$ edges.

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Graph Removal Lemma

For every fixed graph H and every $\epsilon > 0$, there exists $\delta > 0$ such that if G contains at most $\delta n^{v(H)}$ copies of H then G may be made H-free by removing at most ϵn^2 edges.

Removal lemmas

Sparse Graph Removal Lemma [Conlon–Fox–Z.]

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Any subgraph of Γ containing at most $\delta p^{e(H)} n^{v(H)}$ copies of H may be made H-free by removing at most ϵpn^2 edges.

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Regularity lemma for sparse graphs

Definition: (ϵ)-regular

Let G be a graph and X and Y vertex subsets. The induced bipartite graph between X and Y is said to be (ϵ)-regular if

$$|d(U, V) - d(X, Y)| \leq \epsilon p$$

for all $U \subset X$ and $V \subset Y$ with $|U| \geq \epsilon |X|$ and $|V| \geq \epsilon |Y|$, where p is the density of G.

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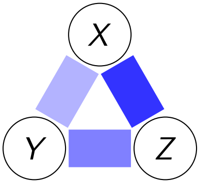
Regularity lemma in sparse graphs (Scott)

For every $\epsilon > 0$ there exists M so that every graph has an (ϵ)-regular partition into at most M parts.

Triangle counting lemma

Setup

- Γ tripartite jumbled graph on vertex sets X, Y, Z.
- G subgraph of Γ, (ϵ)-regular between parts.



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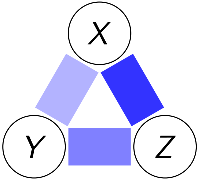

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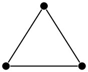
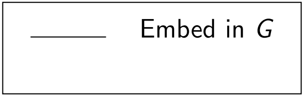
Functional approach to counting





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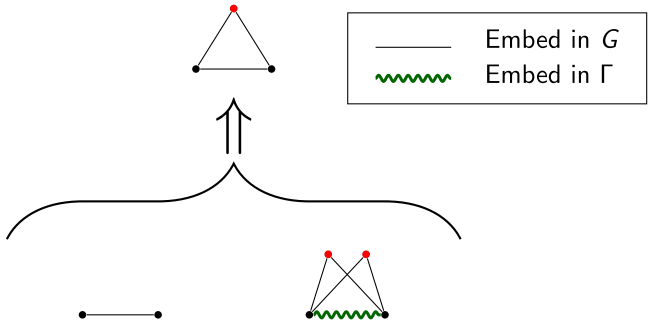

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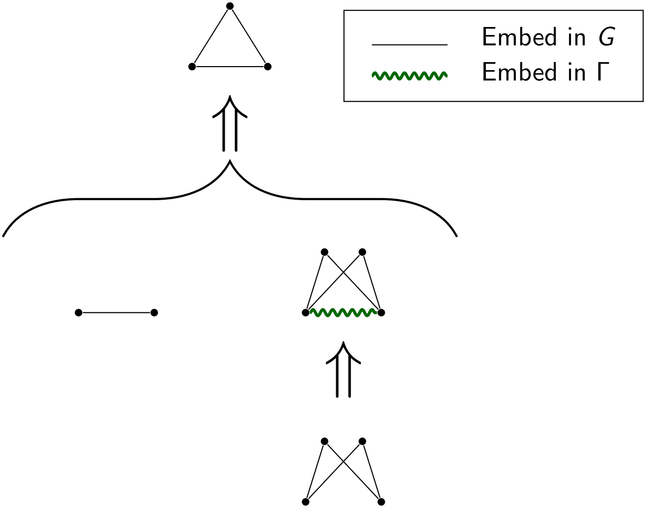

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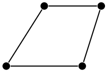

Embed in G

Embed in Γ

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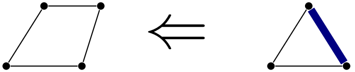

thin line	Embed in G
wavy green line	Embed in Γ
thick blue line	Embed in dense weighted reg. graph

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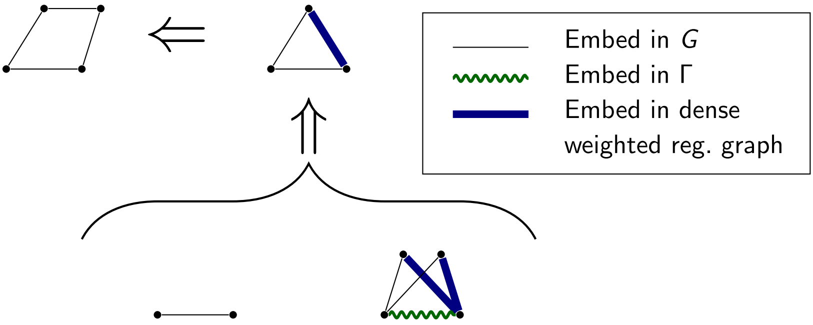

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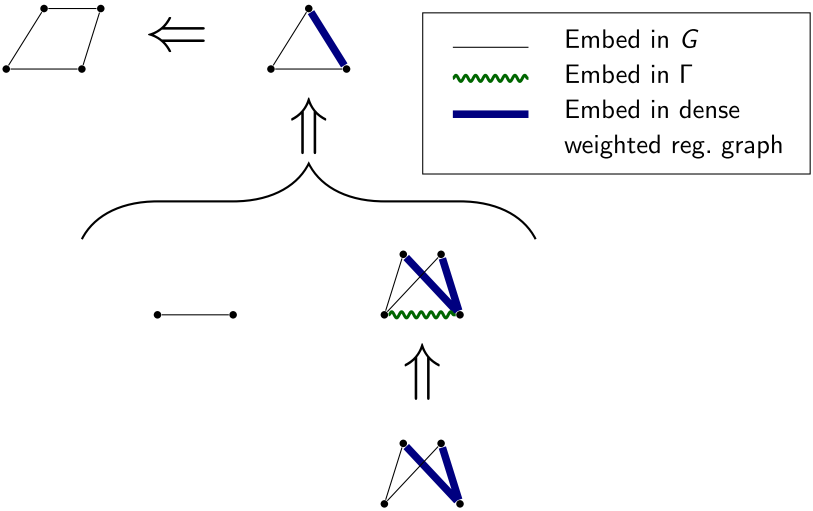

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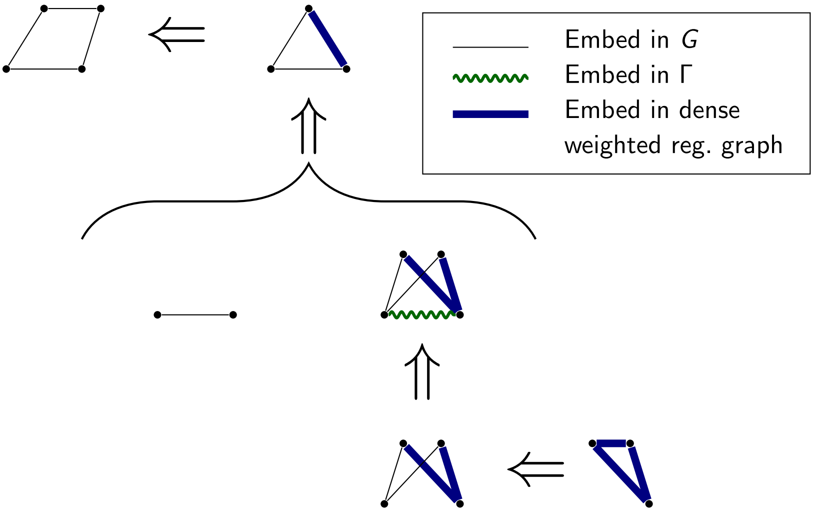

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- Removal lemma, for graphs & groups
- Equivalence of quasirandomness notions
- Induced subgraph counting, induced graph removal lemma
- Improved bounds on induced Ramsey numbers
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