More Sums than Differences Sets and Beyond

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Sum sets and difference sets

For a finite set $S \subset \mathbb{Z}$, let

$$S + S = \{a + b : a, b \in S\}$$
$$S - S = \{a - b : a, b \in S\}$$

Question

Which set is bigger?

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Example: $S = \{0, 1, 3, 8\}$

$S + S = \{0, 1, 2, 3, 4, 6, 8, 9, 11, 16\}$ 10 elements

$S - S = \{-8, -7, -5, -3, -2, -1, 0, 1, 2, 3, 5, 7, 8\}$ 13 elements

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- Since addition is commutative while subtraction is not, two distinct elements generate one sum but two differences.
- So we should expect there to be more differences.

A counterexample

- It was thought that perhaps $|S+S| \leq |S-S|$ for every finite $S \subset \mathbb{Z}$.
- However, in the 1960's, Conway found the following counterexample:

$$S = \{0, 2, 3, 4, 7, 11, 12, 14\}.$$

We have

$$S + S = [0, 28] \setminus \{1, 20, 27\} \qquad \text{26 elements}$$
$$S - S = [-14, 14] \setminus \{-13, -6, 6, 13\} \qquad \text{25 elements}$$

- So it began the search for more of such sets ...

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Later, Nathanson wrote:

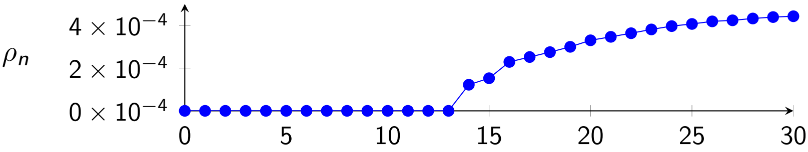
> *A difficult and subtle problem is to decide what is the appropriate method of counting (or, equivalently, the appropriate probability measure) to apply to MSTD sets.*

Proportion of MSTD sets

Theorem (Martin and O'Bryant, 2006)

Let $\rho_n 2^{n+1}$ be the number MSTD subsets of $\{0, 1, \ldots, n\}$. Then $\rho_n \geq 2 \times 10^{-7}$ for $n \geq 14$.

Conjecture: ρ_n has a limit; estimated at 4.5×10^{-4} using Monte Carlo.



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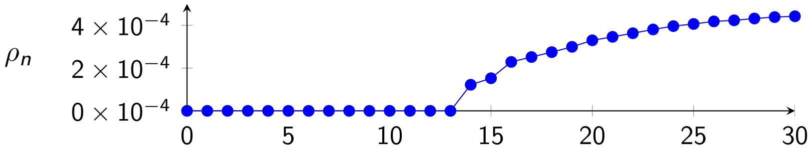

My result

ρ_n converges to a limit $\rho > 4 \times 10^{-4}$.

Furthermore, we have a deterministic algorithm that could, in principle, compute ρ up to arbitrary precision.

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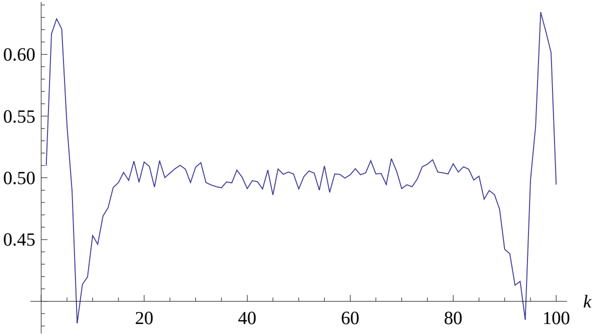
- This intuition helped to prove many results about MSTD sets.
- However, there has been no description on what "most" MSTD looks like.
- We address this question by giving a rigorous formulation of the intuition.

Behavior of the middle portion?

For uniform random subset $S \subset [1, n]$, let

$$\gamma(k, n) = \mathbb{P}(k \in S \mid S \text{ is MSTD})$$

Estimated values of $\gamma(k, 100)$:



Miller et al. [MOS] conjectured that, for any constant $0 < c < 1/2$, if $cn < k < n - cn$, then $\gamma(k, n) \to 1/2$ as $n \to \infty$.

What does a typical MSTD set look like?

Answer: A well-controlled fringe and an almost unrestricted middle.

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Theorem

- $\alpha_n \in \mathbb{Z}$ satisfying $0 < \alpha_n < n/2$ and $\alpha_n \to \infty$ as $n \to \infty$
- S a uniform random subset of $[0, n]$
- E an event that depends only on $S \cap [\alpha_n + 1, n - \alpha_n - 1]$

Then, as $n \to \infty$,

$$|\mathbb{P}(E \mid S \text{ is MSTD}) - \mathbb{P}(E)| = O\left((3/4)^{\alpha_n}\right).$$

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- Central limit theorem: for any $t \in \mathbb{R}$,

$$\mathbb{P}\left(|S| < \frac{n + t\sqrt{n}}{2} \,\middle|\, S \text{ is MSTD}\right) \to \Phi(t)$$

where $\Phi(t)$ is the standard normal distribution.

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- So the size distribution of MSTD sets is very similar to the unrestricted binomial distribution.

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Our method can be modified to give similar answers to each of these questions.

The Number of Missing Sums and Differences

For a subset $S \subset \{0, 1, \dots, n\}$, let

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Let

$$\Lambda \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}.$$

Assume that Λ has at least one element (s, d) with d even. We are interested in

$$\{S \subset \{0, 1, \ldots, n\} : \lambda(S) \in \Lambda\}$$

E.g., $\Lambda = \{(s, d) : s < d\}$ gives MSTD sets

General Results

Let $\Lambda \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ contain at least one element (s, d) with d even.
$\lambda(S) = (2n + 1 - |S + S|, 2n + 1 - |S - S|)$.
Let S be a uniform random subset of $\{0, 1, \ldots, n\}$. As $n \to \infty$,

- $\mathbb{P}(\lambda(S) \in \Lambda)$ approaches some positive limit
- We have a deterministic algorithm for computing this limit up to arbitrary precision
- if $\alpha_n \in \mathbb{Z}$ satisfying $0 < \alpha_n < n/2$ and $\alpha_n \to \infty$, E an event that depends only on $S \cap [\alpha_n + 1, n - \alpha_n - 1]$, and F an event that depends only on $S \cap ([0, \alpha_n] \cup [n - \alpha_n, n])$, then

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