**The joints problem for varieties**

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**Joint work with**

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**Joints problem**

What's the max # of joints that \( N \) lines in \( \mathbb{R}^3 \) can make?

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**A joint** is a point contained in 3 non-coplanar lines.

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**Examples**

1. \( \Theta(N^{3/2}) \) joints
   - \( N^{3/2} \) joints
   - \( \frac{N}{3} \) joints
   - \( \frac{N}{3} \) joints

2. \( k \sim \sqrt{N} \) generic planes
   - pairwise form
   - \( \binom{k}{2} \sim N \) lines
   - triplewise form
   - \( \binom{k}{3} \sim \frac{k^3}{3} N^{3/2} \) joints
Counting and cutting cycles of lines and rods in space*

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Connections

To Kakeya problem in analysis (Wolff)

Finite field Kakeya problem (Dvir) ← polynomial method

Erdős distinct distance problem (Guth-Katz)

Multilinear Kakeya, "joints of tubes" (Bennett-Carbery-Tao, Guth)

Many extensions & generalizations of the joints problem

Kaplan-Sharir-Shustin, Quilodrán Simplification & extension to all dimensions

Thm (joints of lines) $N$ lines in $F^d$ have $O_d(N^{d-1})$ joints.

Joints of flats: max # joints for $N$ planes in $F^d$?

Construction $\Theta(N^{3/2})$ joints: generic 4-flats, $\cap$ pair $\to$ planes, $\cap$ triple $\to$ joints

Why I like this problem:

$\ast$ natural extension of joints

$\ast$ a key step of pf of joints thm fails badly

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*Introduced the joints problem & proved $O(N^{7/4})$ upper bd on # joints

... many subsequent improvements on until:

Algebraic methods in discrete analogs of the Kakeya problem

Larry Guth $^*$, Nets Hawk Katz $^*$$^*$

**Theorem 1.1.** Any set of $N$ lines in $\mathbb{R}^3$ form at most $O(N^{3/2})$ joints.

Recently Hans Yu & I improved the constant factor to optimal: $\leq \frac{12}{3} N^{3/2}$ joints
Incidence geometry for higher dimensional objects

Solymosi-Tao: nearly tight \((t+1)\) in exponent bound for point-variety incidences in \(\mathbb{R}^d\) (in the spirit of Szemerédi-Trotter)

- Extension of the Guth-Katz polynomial partitioning method
  - Use bounded degree polynomials

[Walsh]: a different, algebraic, partitioning method

\(\Rightarrow\) Incidences among higher dim, higher degree varieties in \(\mathbb{F}^d\)

Previous results:

[Yang] \(N\) planes in \(\mathbb{R}^6\) have \(N^{\frac{3}{2}+o(1)}\) joints

(technique: bounded deg partitioning, restrict to codim-1 variety)

Limitations

1. Error term in exponent
2. Only in \(\mathbb{R}\)

[Yu-Z./Carbery-Iliopoulou] \(N\) lines \& \(M\) planes in \(\mathbb{F}^4\) make \(O(NM^{\frac{3}{2}})\) joints

(line-line-plane, in indep \& spanning directions)

Our results

[Tidor-Yu-Z.] \(N\) planes in \(\mathbb{F}^6\) have \(O(N^{\frac{3}{2}})\) joints

\(N\) \(k\)-flats in \(\mathbb{F}^{mk}\) have \(O_{m,k}(N^{\frac{m}{m-1}})\) joints

A set of \(k\)-dim varieties in \(\mathbb{F}^{mk}\) of total degree \(N\) has \(O_{m,k}(N^{\frac{m}{m-1}})\) joints

A new way to apply polynomial method to higher dim objects

Several sets of lines \& flats, varieties \([Conj by Carbery]\)

and count joints formed by taking one object from each set
Multijoints of lines \( (\text{Iliopoulos } \mathbb{R}^d \& \mathbb{F}^3; \text{Zhang } \mathbb{F}^d) \)

\( L_1, L_2, \ldots, L_d \) sets of lines in \( \mathbb{F}^d \)

# joints formed by taking one line from each set is \( \binom{1}{d-1} \sum L_1 \ldots L_d \).

[Tidor-Yu-Z.] We extend from lines to varieties of arb dim.

Joints of lines with multiplicities \( (\text{Carbery conj; Iliopoulos } \mathbb{R}^3; \text{Zhang } \mathbb{F}^d) \)

\( L_1, L_2, \ldots, L_d \) sets of lines in \( \mathbb{F}^d \)

\[
\sum_{\text{joints } p} \left( \#(l_1, \ldots, l_d) \in L_1 \times \cdots \times L_d \text{ making a joint at } p \right)^{\frac{1}{d-1}} \leq \binom{1}{d-1} \sum L_1 \ldots L_d^{\frac{1}{d-1}}.
\]

[Tidor-Yu-Z.] Also \( \text{lines} \rightarrow \text{varieties} \)

Review of the proof of: [Kaplan-Sharir-Shustin, Quilodrán]

\( N \) lines in \( \mathbb{R}^3 \) have \( O(N^{3/2}) \) joints

1. **Parameter counting:**

   Using \( \dim \mathbb{R}[x_1, \ldots, x_n] = (n+d) \)

   deduce that \( \exists \text{non-zero } \text{poly } g, \deg g < C \sqrt{J} \), vanishing on joints

   Take \( g \) with min \( \deg \).

2. **Vanishing lemma:** a single-variable polynomial cannot vanish more times than its degree

3. **A joints-specific argument.** If all lines have \( > C J^{3/2} \) joints,

   then vanishing lemma \( \Rightarrow g \) vanishes on all lines \( \Rightarrow \forall g \) vanishes on all joints

   \( \Rightarrow \) one of \( \partial_x g, \partial_y g, \partial_z g \) is non-zero, lower \( \deg \) & vanish on all joints

   So some line has \( \leq C J^{3/2} \) joints. Remove this line \& induction \( \square \)

How to generalize vanishing lemma to 2-var polynomials? 😃
Thm (Tidor-Yu-Z.) \( N \) planes in \( \mathbb{R}^6 \) have \( O(N^{3/2}) \) joints

Wishful thinking: only if we had something like...

- every nonzero \( g(x,y) \) of deg \( \leq n \) has \( \leq n^2 \) zeros
  
  or

- two polynomials, each deg \( \leq N^{1/6} \), vanishing at all joints and no common factors when restricted to each plane (related: inverse Bézout)

Method of multiplicities: ask a polynomial to vanish at each joint to some high order

By counting parameters, maybe hope for:

- Every nonzero \( g(x,y) \) of deg \( \leq ns \) vanish to order \( \geq s \) at \( \leq n^2 pt \)

Counterexample: \( g(x,y) = y^s \)

Linear dependencies among vanishing conditions

\( e.g.: g(p) = 0, \ xg(p) = 0, (\partial_{xx} - \partial_{xy})g(p) = 0 \)

**Key idea** 1: Restricting to a plane for now

We will construct a set of \( \dim \mathbb{R}[x,y]_{\leq n}(n+2) \)

linearly indep vanishing conditions on \( \mathbb{R}[x,y]_{\leq n} \)
Attached to each point $p$ is a set of vanishing conditions for $g \in \mathbb{R}[x,y]_{\leq n}$:

$$
g(p) = 0, \quad \partial_x g(p) = 0, \quad \partial_y g(p) = 0
$$

$$
\partial_{xx} g(p) = 0, \quad \partial_{xy} g(p) = 0, \quad \partial_{yy} g(p) = 0, \quad \partial_{xxx} g(p) = 0, \ldots
$$

If we take all these conditions up to order $n+1$, then any satisfying $g \in \mathbb{R}[x,y]_{\leq n}$ must be zero.

The above vanishing conditions attached to several different points are linearly dependent as linear functionals on $\mathbb{R}[x,y]_{\leq n}$.

We will select a basis of linear functionals on $\mathbb{R}[x,y]_{\leq n}$ via the following procedure.

**First attempt**

Cycle through the points on the plane (say 100 pts)

$p_1, p_2, p_3, \ldots, p_1, p_2, p_3, \ldots, p_1, p_2, p_3, \ldots$

$p_1$ : add vanishing condition $g(p_1) = 0$

$p_2$ : add vanishing condition $g(p_2) = 0$, as long as it does not already follow from previously added vanishing condition.

$p_3$ : add a nonredundant subset of $\partial_x g(p_1) = 0, \partial_y g(p_1) = 0$

none implied by other added + prev. added

i.e. basis extension

$p_2$ : add a nonredundant subset of $\partial_x g(p_2) = 0, \partial_y g(p_2) = 0$
The process assigns a total of \( \binom{n+2}{2} \) vanishing conditions each attached to a point.

Can we control the # of vanishing attached to each pt?

Example:

![Diagram]

\( \text{pts on grid get way more vanishing conditions than pts on the line} \)

-Un-desirable

(This example also comes up for inverse Bézout; see Tao blog)

\*Key idea 2\* Let some points get a head start

\( e.g. \quad P_1, P_2, \ldots, P_{50}, \quad P_1, \ldots, P_{50}, \quad \ldots, P_1, \ldots, P_{100}, \quad P_1, \ldots, P_{100}, \ldots \)

Handicap \( \vec{d} \) assigns an integer to each point

\( e.g. \quad \begin{array}{cccccc}
\text{points} & a & b & c & d & e \\
\text{handicap} & 0 & 1 & 3 & 0 & -1
\end{array} \)

\( \rightarrow \text{order: c c b c a b c d a b c d e a b c d e ...} \)

Modify process of assigning vanishing conditions

\( c \) : add a non-redundant set of 0th order derivative vanishing @c

\( c \) : 

\( \begin{array}{cccccc}
\text{1st} & \ldots & \ldots & \ldots & \ldots & \ldots \\
\text{2nd} & \ldots & \ldots & \ldots & \ldots & \ldots \\
\text{3rd} & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array} \)
Want a “good” choice of handicaps: treating all joints “fairly”

Hard to compute how # vanishing cond at each pt depends on the handicap.

1. Monotonicity: \( \alpha_p \Rightarrow \# \text{van. cond at } p \text{ cannot } \Delta \)
2. Lipschitz continuity: small \( \Delta \) in handicap \( \Rightarrow \) small \( \Delta \) in \( \# \text{van cond} \)
3. Bounded domain: suffices to consider handicaps with bounded values (else some pt gets 0 van. cond)

Key idea 3: Decide on handicap choice later, implicitly (existence via compactness/smoothing)

Putting different planes together at joints of planes in \( \mathbb{R}^b \)

Handicap \( \alpha \in \mathbb{Z}^g \) assigns an integer to each joint

Separately for each plane \( F \), apply above process to assign vanishing conditions restricted to \( F \) to joints on \( F \)

A new vanishing lemma: Given \( 0 \neq g \in \mathbb{R}[x_1, \ldots, x_b] \leq \eta \),

\( \exists \) joint \( p \), contained in planes \( F_1, F_2, F_3 \) (indepl spanning directions) \& derivative operator \( D \), assigned to \( p \) on \( F_1 \) (likewise \( D_2, D_3 \))

s.t. \( D_1D_2D_3 g(p) \neq 0 \).
By parameter counting, 
\[ \sum_{\text{joint } p} \left( \text{# choices of } D_1, D_2, D_3 \text{ at } p \right) \geq \dim \mathbb{R}[x, \ldots, x_6]_{\leq n} = \binom{n+6}{6} \]

By compactness/smoothing, \( \exists \) handicap so that these terms are roughly all equal.

Also recall that \( \# \) van. cond. on each plane is exactly \( \binom{n+2}{2} \)

Putting together + AM-AM \( \Rightarrow \) theorem

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**Joints of varieties**

**Flats**: higher order directional directives along a flat.

**Varieties**: derivatives in local coordinates.

E.g. \( y = x^2 + y^2 \) on the circle,

\[ y = x^2 + y^2 = x^2 + (x^2 + y^2)^2 = x^2 + x^4 + 2x^6 + \cdots \]

2nd order derivative operator at the origin is \( \frac{\partial^2}{\partial x^2} + \frac{2}{\partial y} \) (not \( \frac{\partial^2}{\partial x^2} \)) so that evaluations give linear functional on the space of regular functions.

**Extension to arbitrary fields \( F \)**

When differentiating, we only care about coeff extraction.

**Hasse derivatives** (formal algebraic derivatives).

**Question**: Other applications of this variant of polynomial method for higher dim objects?