The Number of Independent Sets in a Regular Graph

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Introduction

Let $G = (V, E)$ be a graph. An **independent set** is a subset of the vertices with no two adjacent. Let $i(G)$ denote the number of independent sets of G.

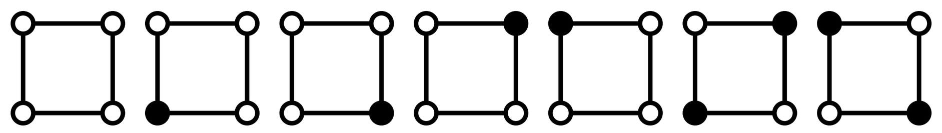


Figure 1: *The independent sets of a 4-cycle:* $i(C_4) = 7$.

The following question is motivated by applications in combinatorial group theory [1] and statistical mechanics [4].

Question. In the family of N-vertex, d-regular graphs, when is the number of independent sets maximized?

Alon [1] in 1991 and Kahn [4] in 2001 conjectured that, when $N/2d \in \mathbf{Z}$, $i(G)$ should be maximized when G is a disjoint union of $N/2d$ copies of $K_{d,d}$, which has $i(K_{d,d})^{N/2d}$ independent sets since $i(G_1 \sqcup G_2) = i(G_1)i(G_2)$ for any graphs G_1 and G_2. More precisely, it was conjectured that:

Conjecture (Alon and Kahn)**.** For any N-vertex, d-regular graph G,

$$i(G) \le i(K_{d,d})^{N/2d} = \left(2^{d+1} - 1\right)^{N/2d}.$$

Note equality holds if G is a disjoint union of $K_{d,d}$'s.

Our result confirms and generalizes this conjecture.

Example: Two 6-vertex 3-regular graphs:

13 independent sets 15 independent sets

Previous results

Alon [1]	$i(G) \le 2^{(1/2+O(d^{-0.1}))N}$
Kahn [4]	Proved conjecture for bipartite G
Sapozhenko [6]	$i(G) \le 2^{(1/2+O(\sqrt{(\log d)/d}))N}$
Kahn [5]	$i(G) \le 2^{(1/2+1/d)N}$
Galvin [2]	$i(G) \le 2^{(1/2+1/2d+O(\sqrt{(\log d)/d^3}))N}$

Main Result

For any N-vertex, d-regular graph G, and any $\lambda \ge 0$,

$$P(\lambda, G) \le P(\lambda, K_{d,d})^{N/2d} = \left(2(1+\lambda)^d - 1\right)^{N/2d},$$

with equality if G is a disjoint union of $K_{d,d}$'s. Here

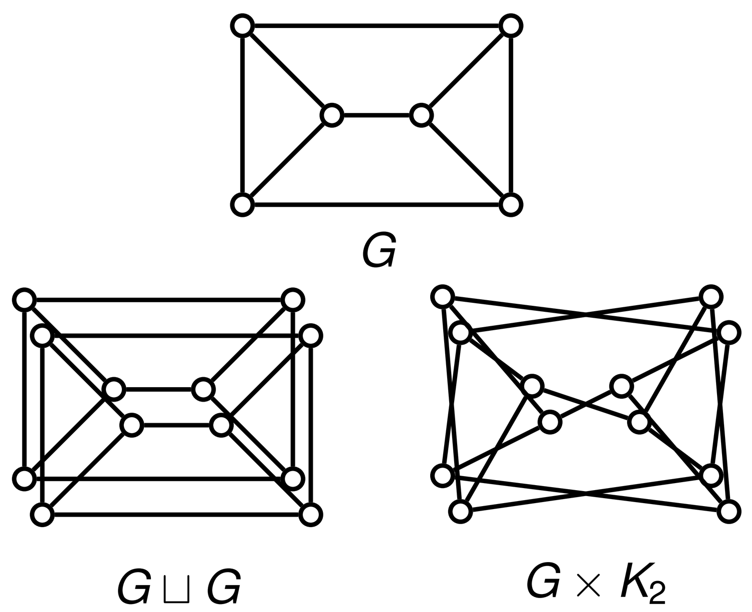
$$P(\lambda, G) = \sum_{I \text{ ind. set}} \lambda^{|I|} = \sum_{k \ge 0} (\#\text{ ind. sets of size } k)\lambda^k.$$

Setting $\lambda = 1$ yields the Alon-Kahn conjecture.

Proof

We prove our main result by reducing it to the bipartite case, which was proven by Galvin and Tetali [3] (and by Kahn [4] for the non-weighted case).

From G we build $G \sqcup G$ and $G \times K_2$:



Idea. Show that $G \times K_2$ has at least as many independent sets of each size as $G \sqcup G$.

This would imply that, for $\lambda \ge 0$,

$$\begin{aligned} P(\lambda, G \sqcup G) &= \sum_{k\ge 0} (\#\text{ ind. sets of size } k \text{ in } G \sqcup G)\lambda^k \\ &\le \sum_{k\ge 0} (\#\text{ ind. sets of size } k \text{ in } G \times K_2)\lambda^k \\ &= P(\lambda, G \times K_2). \end{aligned}$$

Note that $P(\lambda, G \sqcup G) = P(\lambda, G)^2$ since independent sets of $G \sqcup G$ correspond to pairs of independent sets of G. The main result holds for $G \times K_2$ since it's already bipartite. So

$$P(\lambda, G)^2 = P(\lambda, G \sqcup G) \le P(\lambda, G \times K_2) \le P(\lambda, K_{d,d})^{N/d},$$

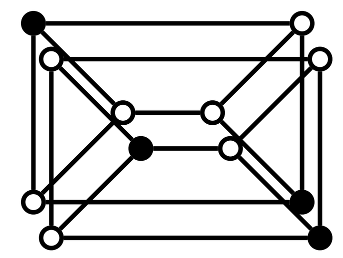
from which the result for G would follow. So we have reduced the problem to the lemma on the next column.

Key Lemma

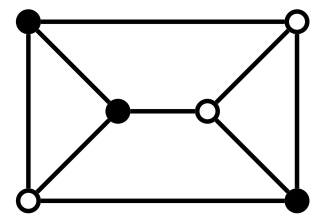
For any graph G, there exists a size-preserving injection from $\mathcal{I}(G \sqcup G)$ to $\mathcal{I}(G \times K_2)$, where $\mathcal{I}(\cdot)$ denotes the collection of independent sets of a graph.

Construction of the injection:

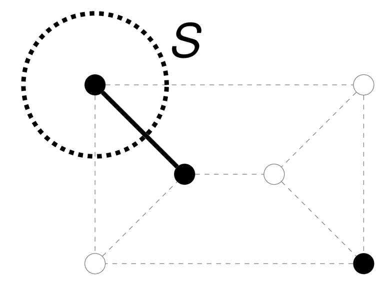
- Start with an independent set $A \sqcup B$ of $G \sqcup G$:



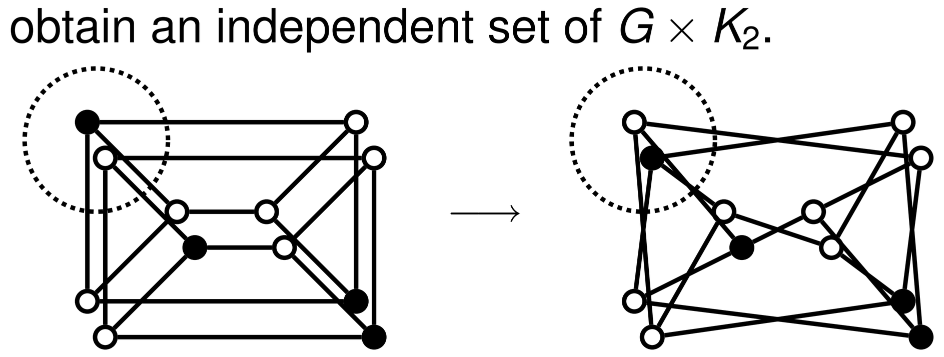
- "Merge" the two layers. Obtain $A \cup B \subset V(G)$.



- The induced subgraph $G[A \cup B]$ is a bipartite graph since it is induced by the union of two independent sets. Choose the lexicographically first $S \subset V(G)$ so that all edges of $G[A \cup B]$ lie between S and $V(G) \setminus S$.



- Back to $G \sqcup G$. Swap each pair of vertices in S, and we obtain an independent set of $G \times K_2$.



Claim. This is an injection whose image consists of all independent sets $C \sqcup D$ of $G \times K_2$ such that $G[C \cup D]$ is bipartite. Here $C, D \subset V$ correspond to the two "layers" of $G \times K_2$.

Proof. The construction always produces an independent set of $G \times K_2$ since swapping the vertices of S eliminates all possible adjacencies in $G \times K_2$.

We obtain the inverse map by basically the same procedure. See [7] for details. □

Further Questions

Non-regular graphs. Kahn [4] also conjectured that, for any graph G without isolated vertices

$$i(G) \le \prod_{uv \in E(G)} \left(2^{\deg(u)} + 2^{\deg(v)} - 1\right)^{1/\deg(u)\deg(v)}.$$

Non-entropy proof of bipartite case? So far the only known proofs of the bipartite case of these results use entropy methods [3, 4]. It would be nice to have an elementary and completely combinatorial proof.

Counting graph homomorphisms. Galvin and Tetali [3] generalized Kahn's result and showed that for any d-regular, N-vertex bipartite graph G, and any graph H (possibly with self-loops),

$$|\mathrm{Hom}(G, H)| \le |\mathrm{Hom}(K_{d,d}, H)|^{N/2d},$$

Graph homomorphisms generalize the notion of independent sets as well as colorings. It is suspected that the inequality holds also for non-bipartite G as long as H is "nice", but we do not have a proof.

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References

[1] N. Alon, *Independent sets in regular graphs and sum-free subsets of finite groups*, Israel J. Math. **73** (1991), no. 2, 247–256.

[2] D. Galvin, *An upper bound for the number of independent sets in regular graphs*, Discrete Math. (to appear).

[3] D. Galvin and P. Tetali, *On weighted graph homomorphisms*, Graphs, morphisms and statistical physics, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., vol. 63, Amer. Math. Soc., Providence, RI, 2004, pp. 97–104.

[4] J. Kahn, *An entropy approach to the hard-core model on bipartite graphs*, Combin. Probab. Comput. **10** (2001), no. 3, 219–237.

[5] M. Madiman and P. Tetali, *Information inequalities for joint distributions, with interpretations and applications*, IEEE Trans. on Information Theory (to appear).

[6] A. A. Sapozhenko, *On the number of independent sets in extenders*, Diskret. Mat. **13** (2001), no. 1, 56–62.

[7] Y. Zhao, *The number of independent sets in a regular graph*, Combin. Probab. Comput. (to appear).